

# “Algebraic Truths” *vs* “Geometric Fantasies”: Weierstrass’ Response to Riemann

U. Bottazzini\*

## Abstract

In the 1850s Weierstrass succeeded in solving the Jacobi inversion problem for the hyper-elliptic case, and claimed he was able to solve the general problem. At about the same time Riemann successfully applied the geometric methods that he set up in his thesis (1851) to the study of Abelian integrals, and the solution of Jacobi inversion problem. In response to Riemann’s achievements, by the early 1860s Weierstrass began to build the theory of analytic functions in a systematic way on arithmetical foundations, and to present it in his lectures. According to Weierstrass, this theory provided the foundations of the whole of both elliptic and Abelian function theory, the latter being the ultimate goal of his mathematical work. Riemann’s theory of complex functions seems to have been the background of Weierstrass’s work and lectures. Weierstrass’ unpublished correspondence with his former student Schwarz provides strong evidence of this. Many of Weierstrass’ results, including his example of a continuous non-differentiable function as well as his counter-example to Dirichlet principle, were motivated by his criticism of Riemann’s methods, and his distrust in Riemann’s “geometric fantasies”. Instead, he chose the power series approach because of his conviction that the theory of analytic functions had to be founded on simple “algebraic truths”. Even though Weierstrass failed to build a satisfactory theory of functions of several complex variables, the contradiction between his and Riemann’s geometric approach remained effective until the early decades of the 20<sup>th</sup> century.

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## Introduction

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\*Dipartimento di matematica, Università di Palermo, via Archirafi 34, 90123 Palermo, Italy.  
 E-mail: bottazzi@math.unipa.it

In 1854 Crelle's *Journal* published a paper on Abelian functions by an unknown school teacher. This paper announced the entry in the mathematical world of a major figure, Karl Weierstrass (1815-1897), who was to dominate the scene for the next forty years to come. His paper presented a solution of Jacobi inversion problem in the hyper-elliptic case. In analogy with the inversion of elliptic integrals of the first kind, Jacobi unsuccessfully attempted a direct inversion of a hyper-elliptic integral of the first kind. This led him to consider multi-valued, "unreasonable" functions having a "strong multiplicity" of periods, including periods of arbitrarily small (non-zero) absolute value. Jacobi confessed he was "almost in despair" about the possibility of the inversion when he realized "by divination" that Abel's theorem provided him with the key for resurrecting the analogy with the inversion of elliptic integrals by considering the sum of a suitable number of (linearly independent) hyper-elliptic integrals instead of a single integral. In his memoir submitted to the Paris Academy in 1826 (and published only in 1841) Abel had stated a theorem which extended Euler's addition theorem for elliptic integrals to more general (Abelian) integrals of the form  $\int R(x, y)dx$  in which  $R(x, y)$  is a rational function and  $y = y(x)$  is an algebraic function defined by a (irreducible) polynomial equation  $f(x, y) = 0$ . According to Abel's theorem, the sum of any number of such integrals reduces to the sum of a number  $p$  of linearly independent integrals and of an algebraic-logarithmic expression ( $p$  was later called by Clebsch the genus of the algebraic curve  $f(x, y) = 0$ ). In 1828 Abel published an excerpt of his Paris memoir dealing with the particular (hyper-elliptic) case of the theorem, when  $f(x, y) = y^2 - P(x)$ ,  $P$  is a polynomial of degree  $n > 4$  having no multiple roots. In this case  $p = [(n - 1)/2]$ , and for hyper-elliptic integrals of the first kind  $\int \frac{Q(x)dx}{\sqrt{P(x)}}$  ( $Q$  is a polynomial of degree  $\leq p - 1$ ) the algebraic-logarithmic expression vanishes ([1], vol. 1, 444-456).

On the basis of Abel's theorem in 1832 Jacobi formulated the problem of investigating the inversion of a system of  $p$  hyper-elliptic integrals

$$u_k = \sum_{j=0}^{p-1} \int_0^{x_j} \frac{x^k dx}{\sqrt{P(x)}} \quad (0 \leq k \leq p-1) \quad (\deg P = 2p+1 \text{ or } 2p+2)$$

by studying  $x_0, x_1, \dots, x_{p-1}$  as functions of the variables  $u_0, u_1, \dots, u_{p-1}$ . These functions  $x_i = \lambda_i(u_0, u_1, \dots, u_{p-1})$  generalized the elliptic functions to  $2p$ -periodic functions of  $p$  variables. Jacobi's "general theorem" claimed that  $x_0, x_1, \dots, x_{p-1}$  were the roots of an algebraic equation of degree  $p$  whose coefficient were single-valued,  $2p$ -periodic functions of  $u_0, u_1, \dots, u_{p-1}$ . Therefore, the elementary symmetric functions of  $x_0, x_1, \dots, x_{p-1}$  could be expressed by means of single-valued functions in  $C^p$ . In particular, Jacobi considered the case  $p = 2$  ([4], vol. 2, 7-16). His ideas were successfully developed by A. Göpel in 1847 (and, independently of him, J. G. Rosenhain in 1851). The required 4-fold periodic functions of two complex variables were expressed as the ratio of two  $\theta$ -series of two complex variables obtained by a direct and cumbersome computation. This involved an impressive amount of calculations and could hardly be extended to the case  $p > 2$ .

Following a completely different route Weierstrass was able to solve the problem for any  $p$ . Because of his achievements he was awarded a doctor degree *honoris causa* from the Königsberg University, and two years later he was hired to teach at the Berlin Gewerbeinstitut (later Gewerbeakademie, today Technische Universität). Eventually, in the Fall of 1856 Weierstrass was named *Extraordinarius* at the Berlin University.

## 1. Weierstrass' early papers

In the address he gave in 1857 upon entering the Berlin Academy, Weierstrass recognized the “powerful attraction” which the theory of elliptic functions had exerted on him since his student days. In order to become a school teacher in 1839 Weierstrass had entered the Theological and Philosophical Academy of Münster, where he attended for one semester Gudermann's lectures on elliptic functions and became familiar with the concept of uniform convergence which Gudermann had introduced in his papers in 1838. Elliptic functions constituted the subject of Weierstrass' very first paper, an essay he wrote in autumn 1840 for obtaining his *venia docendi*. His starting point was Abel's claim that the elliptic function which is the inverse of the elliptic integral of first kind ( $\text{snu}$  in the symbolism Weierstrass took from Gudermann) could be expressed as the ratio of two convergent power series of  $u$ , whose coefficients are entire functions of the modulus of the integral. Weierstrass succeeded in proving that  $\text{snu}$  (and similarly  $\text{cnu}$  and  $\text{dnu}$ ) could be represented as quotient of certain functions, which he named *Al*-functions in honor of Abel and which he was able to expand in convergent power series.

Working in complete isolation without any knowledge of Cauchy's related results, two years before Laurent, Weierstrass (1841) succeeded in establishing the Laurent expansion of a function in an annulus. In the paper he made an essential use of integrals, and proved the Cauchy integral theorem for annuli ([10], vol. 1, 51-66). In a subsequent paper he stated and proved three theorems on power series. Theorems A) and B) provided estimates (Cauchy inequalities) for the coefficients of a Laurent series in one (and several) complex variables, while Theorem C) was the double series theorem nowadays called after him. As a consequence of it Weierstrass obtained the theorem on uniform differentiation of convergent series ([10], vol. 1, 67-74). Apparently, this paper marked a turning point in Weierstrass' analytic methods for he gave up integrals and choose the power series approach to treat the theory of function of one or more variables on a par. This work was completed by a paper he wrote in spring 1842. There Weierstrass proved that a system of  $n$  differential equations

$$\frac{dx}{dt} = G_i(x_1, \dots, x_n) \quad (i = 1, \dots, n) \quad (G_i(x_1, \dots, x_n) \text{ polynomials})$$

can be solved by a system of  $n$  unconditionally and uniformly convergent power series satisfying prescribed initial conditions for  $t=0$ . In addition, he also showed how the power series

$$x_i = P_i(t - t_0, a_1, \dots, a_n) \quad (i = 1, \dots, n; t_0, a_1, \dots, a_n \text{ fixed})$$

convergent in a disk centered at  $t_0$  could be analytically continued outside the disk. Thus, by the early 1840s the essential results of Weierstrass' approach to the theory of analytic functions were already established. His papers, however, remained in manuscript and had no influence on the contemporary development of mathematics.

## 2. Abelian functions and integrals

Weierstrass' 1854 paper ([10], vol. 1, 133-152) gave "a short overview" of the work on Abelian functions which he had developed "several years ago" and summarized in the annual report of the Braunsberg Gymnasium for 1848-49. Weierstrass began by considering the polynomial  $R(x) = (x - a_0)(x - a_1) \cdots (x - a_{2n})$ , with  $a_i$  real numbers satisfying the inequalities  $a_i > a_{i+1}$ . He decomposed  $R(x)$  into the factors  $P(x) = \prod_{k=1}^n (x - a_{2k-1})$ ,  $Q(x) = \prod_{k=1}^n (x - a_{2k})$  and considered the system

$$u_m = \sum_{j=1}^n \int_{a_{2j-1}}^{x_j} \frac{P(x)}{x - a_{2m-1}} \frac{dx}{2\sqrt{R(x)}} \quad (m = 1, \dots, n). \quad (2.1)$$

The task Weierstrass gave himself was to "establish in detail" Jacobi's theorem which he considered "the foundations of the whole theory". As Jacobi had remarked, for given values of  $x_1, x_2, \dots, x_n$  the quantities  $u_1, u_2, \dots, u_n$  have infinitely many different values. "Conversely, if the values of  $u_1, u_2, \dots, u_n$  are given, then the values of  $x_1, x_2, \dots, x_n$  as well as the corresponding values of  $\sqrt{R(x_1)}, \sqrt{R(x_2)}, \dots, \sqrt{R(x_n)}$  are uniquely determined". Moreover, " $x_1, x_2, \dots, x_n$  are roots of a (polynomial) equation of degree  $n$  whose coefficient are completely determined, single-valued functions of the variables  $u_1, u_2, \dots, u_n$ ". Analogously, Weierstrass added, there exists a polynomial function of  $x$ , whose coefficients are also single-valued functions of  $u_1, u_2, \dots, u_n$  which gives the corresponding values of  $\sqrt{R(x_1)}, \sqrt{R(x_2)}, \dots, \sqrt{R(x_n)}$  for  $x = x_1, x_2, \dots, x_n$ . Every rational symmetric function of  $x_1, x_2, \dots, x_n$  could consequently be regarded as a single-valued function of  $u_1, u_2, \dots, u_n$ . Weierstrass considered the product  $L(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$  and the  $2n + 1$  single-valued functions  $Al(u_1, u_2, u_n)_m = \sqrt{h_m L(a_m)}$  ( $m = 0, \dots, 2n + 1$ ), with  $h_m$  suitable constants, which he called Abelian functions, "since they are the ones which completely correspond to the elliptic functions" to which they reduce when  $n = 1$ .

He was able to expand his  $Al$ -functions in convergent power series and, on the basis of Abel's theorem, he succeeded in establish the "principal property" of such functions, i.e. an addition theorem according to which  $Al(u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)_m$  are rationally expressed in terms of  $Al(u_1, u_2, \dots, u_n)_m$ ,  $Al(v_1, v_2, \dots, v_n)_m$  and their first-order partial derivatives. Eventually, he determined the algebraic equation whose coefficients were expressed in terms of  $Al$ -functions, and whose roots were the quantities  $x_1, x_2, \dots, x_n$  satisfying equations 2.1) for arbitrary  $u_1, u_2, \dots, u_n$ . However, as Dirichlet commented, in his paper Weierstrass "gave only partial proofs of his results and lacked the intermediate explanations" ([3], 52).

Two years later Weierstrass resumed this work and published in Crelle's *Journal* the first part of an expanded and detailed version of it ([10], vol. 1, 297-355).

As he had done in his 1854 paper, Weierstrass considered the polynomial  $R(x) = A(x - a_1)(x - a_2) \cdots (x - a_{2\rho+1})$ , and the analogous product  $P(x) = \prod_{j=1}^{\rho} (x - a_j)$ , ( $j = 1, \dots, \rho$ ) where this time the  $a_j$  were any complex numbers such that  $a_j \neq a_k$  for  $j \neq k$ . Instead of equations 2.1) he considered the corresponding system of differential equations

$$du_m = \sum_{j=1}^{\rho} \frac{1}{2} \frac{P(x_j)}{x_j - a_m} \frac{dx}{\sqrt{R(x_j)}} \quad (m = 1, \dots, \rho) \quad (2.2)$$

and formulated Jacobi inversion problem as the question to find solutions  $x_j = x_j(u_1, \dots, u_{\rho})$  of the system 2.2) satisfying the initial conditions  $x_j(0, \dots, 0) = a_j$  ( $j = 1, \dots, \rho$ ). In order to obtain the elliptic functions as a special case ( $\rho = 1$ ), he also gave a slightly different form to his  $Al$ -functions with respect to his previous paper. Weierstrass succeeded in proving that the solutions  $x_j = x_j(u_1, \dots, u_{\rho})$  are single-valued functions of  $u_1, \dots, u_{\rho}$  in the neighborhood of the origin. They could be considered as the roots of a polynomial equation of degree  $\rho$ , whose coefficients were given in terms of  $Al$ -functions which, for any bounded value of  $(u_1, \dots, u_{\rho})$ , are single-valued functions expressed as quotient of power series. Then, the symmetric functions of  $x_j = x_j(u_1, \dots, u_{\rho})$  have “the character of rational functions”. From these results, however, Weierstrass was unable to show that each Abelian function could be represented as the ratio of two everywhere convergent power series. “Here we encounter a problem that, as far as I know, has not yet been studied in its general form, but is nevertheless of particular importance for the theory of functions” ([10], vol. 1, 347).

In the course of his life he returned many times to this problem in an attempt to solve it (see below). Even the factorization theorem for entire functions that Weierstrass was able to establish some 20 years later (see Section 5) can be regarded as an outcome of this research for it provided a positive answer to the problem in the case of one variable ([5], 247). In order to show that his approach permitted one to treat the theory of elliptic and Abelian function on a par, in the concluding part of his 1856 paper Weierstrass presented a detour on elliptic functions, where he summarized the main results he had obtained in 1840. However, the promised continuation of Weierstrass’ paper never appeared. Instead, a completely new approach to the theory of Abelian integrals was published by Bernhard Riemann (1826-1866) in 1857 ([7], 88-144) which surpassed by far anything Weierstrass had been able to produce.

In the introductory paragraphs of his paper Riemann summarized the geometric approach to complex function theory he had set up in his 1851 thesis ([7], 3-48). There he defined a complex variable  $w$  as a function of  $x + iy$  when  $w$  varies according to the equation  $i \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y}$  “without assuming an expression of  $w$  in terms  $x$  and  $y$ ”. Accordingly, “by a well known theorem” - Riemann observed without mentioning Cauchy - a function  $w$  can be expanded in a power series  $\sum a_n(z - a)^n$  in a suitable disk and “can be continued analytically outside it in only one way” ([7], 88). For dealing with multi-valued functions such as algebraic functions and their integrals Riemann introduced one of his deepest achievements, the idea of representing the branches of a function by a surface multiply covering the complex

plane (or the Riemann sphere). Thus, “the multi-valued function has only *one* value defined at each point of such a surface representing its branching, and can therefore be regarded as a completely determined (=single-valued) function of position on this surface” ([7], 91). Having introduced such basic topological concepts as cross-cuts and order of connectivity of a surface, Riemann could state the fundamental existence theorem of a complex function on the surface, which he had proved in his dissertation by means of a suitable generalization of the Dirichlet principle. This theorem establishes the existence of a complex function in terms of boundary conditions and behavior of the function at the branch-points and singularities. Then Riemann developed the theory of Abelian functions proper. It is worth noting that, in spite of the fact that both Weierstrass and Riemann gave their paper the same title and used the same wording, they gave it a different meaning. Whereas the former defined Abelian functions to be the single-valued, analytic functions of several complex variables related to his solution of the Jacobi inversion problem, the latter understood Abelian functions to be the integrals of algebraic functions introduced by Abel’s theorem. In the first part of his paper Riemann developed a general theory of such functions and integrals on a surface of any genus  $p$ , “insofar as this does not depend on the consideration of  $\theta$ -series” ([7], 100). He was able to classify Abelian functions (integrals) into three classes according to their singularities, to determine the meromorphic functions on a surface, and to formulate Abel’s theorem in new terms, thus throwing new light on the geometric theory of birational transformations. The second part of the paper was devoted to the study of  $\theta$ -series of  $p$  complex variables, which express “the Jacobi inverse functions of  $p$  variables for an arbitrary system of finite integrals of equiramified,  $(2p + 1)$ -connected algebraic functions” ([7], 101). In this part Riemann gave a complete solution of Jacobi inversion problem without stating it as a special result. He regarded the work of Weierstrass as a particular case, and mentioned the “beautiful results” contained in the latter’s 1856 paper, whose continuation could show “how much their results and their methods coincided”. However, after the publication of Riemann’s paper Weierstrass decided to withdraw the continuation of his own. Even though Riemann’s work “was based on foundations completely different from mine, one can immediately recognize that his results coincide completely with mine” Weierstrass later stated ([10], vol. 4, 9-10). “The proof of this requires some research of algebraic nature”. By the end of 1869 he had not been able to overcome all the related “algebraic difficulties”. Yet, Weierstrass thought he had succeeded in finding the way to represent any single-valued  $2p$ -periodic (meromorphic) function as the ratio of two suitable  $\theta$ -series, thus solving the general inversion problem. However, Weierstrass’ paper ([10], vol. 2, 45-48) was flawed by some inaccuracies that he himself later recognized in a letter to Borchardt in 1879 (*ibid.*, 125-133). In particular, Weierstrass (mistakenly) stated that any domain of  $C^n$  is the natural domain of existence of a meromorphic function. (This mistake was to be pointed out in papers by F. Hartogs and E.E. Levi in the first decade of the 20<sup>th</sup> century). By 1857, in his address to the Berlin Academy Weierstrass limited himself to state that “one of the main problems of mathematics” which he decided to investigate was “to give an actual representation” of Abelian functions. He recognized that he had

published results “in an incomplete form”. “However - Weierstrass continued - it would be foolish if I were to try to think only about solving such a problem, without being prepared by a deep study of the methods that I am to use and without first practicing on the solution of less difficult problems” ([10], vol. 1, 224). The realization of this program became the scope of his University lectures.

### 3. Weierstrass’ lectures

In response to Riemann’s achievements, Weierstrass devoted himself to “a deep study of the methods” of the theory of analytic functions which in his view provided the foundations of the whole building of the theory of both elliptic and Abelian functions. As Poincaré once stated, Weierstrass’ work could be summarized as follows: 1) To develop the general theory of functions, of one, two and several variables. This was “the basis on which the whole pyramid should be built”. 2) To improve the theory of the elliptic functions and to put them into a form which could be easily generalized to their “natural extension”, the Abelian functions. 3) Eventually, to tackle the Abelian functions themselves.

Over the years the aim of establishing the foundations of analytic function theory with absolute rigor on an arithmetic basis became one of Weierstrass’ major concerns. From the mid-1860s to the end of his teaching career Weierstrass used to present the whole of analysis in a two-year lecture cycle as follows:

1. Introduction to analytic function theory,
2. Elliptic functions,
3. Abelian functions,
4. Applications of elliptic functions or, alternatively, Calculus of variations.

All of these lectures, except for the introduction to analytic function theory, have been published in Weierstrass’ *Werke*. For some twenty years he worked out his theory of analytic functions through continuous refinements and improvements, without deciding to publish it himself. Weierstrass used to present his discoveries in his lectures, and only occasionally communicated them to the Berlin Academy. This attitude, combined with his dislike of publishing his results in printed papers and the fact that he discouraged his students from publishing lecture notes of his courses, eventually gave Weierstrass’ lectures an aura of uniqueness and exceptionality.

### 4. Conversations in Berlin

In the Fall of 1864, when Riemann was staying in Pisa because of his poor health conditions, the Italian mathematician F. Casorati travelled to Berlin to meet Weierstrass and his colleagues. Rumors about new discoveries made by Weierstrass, combined with lack of publications, motivated Casorati’s journey.

“Riemann’s things are creating difficulties in Berlin”, Casorati recorded in his notes. Kronecker claimed that “mathematicians . . . are a bit arrogant (*hochmütig*) in using the concept of function”. Referring to Riemann’s proof of the Dirichlet

principle, Kronecker remarked that Riemann himself, “who is generally very precise, is not beyond censure in this regard” ([2], 262).

Kronecker added that in Riemann’s paper on Abelian functions the  $\theta$ -series in several variables “came out of the blue”. Weierstrass claimed that “he understood Riemann, because he already possessed the results of his [Riemann’s] research”. As for Riemann surfaces, they were nothing other than “geometric fantasies”. According to Weierstrass, “Riemann’s disciples are making the mistake of attributing everything to their master, while many [discoveries] had already been made by and are due to Cauchy, etc.; Riemann did nothing more than to dress them in his manner for his convenience”. Analytic continuation was a case in point. Riemann had referred to it in various places but, in Weierstrass’s and Kronecker’s opinion, nowhere he had treated it with the necessary rigor. Weierstrass observed that Riemann apparently shared the idea that it is always possible to continue a function to any point of the complex plane along a path that avoids critical points (branch-points, and singularities). “But this is not possible”, Weierstrass added. “It was precisely while searching for the proof of the general possibility that he realized it was in general impossible”. Kronecker provided Casorati with the example of the (lacunary) series

$$\theta_0(q) = 1 + 2 \sum_{n \geq 1} q^{n^2} \quad (4.1)$$

which is convergent for  $|q| < 1$ , and has the unit circle as a natural boundary. Its unit circle is “entirely made of points where the function is not defined, it can take any value there”, Weierstrass observed. He had believed that points in which a function “ceases to be definite” - as was the case of the function  $e^{1/x}$  at  $x = 0$  because “it can have any possible value” there - “could not form a continuum, and consequently that there is at least one point  $P$  where one can always pass from one closed portion of the plane to any other point of it”.  $\theta_0(q)$  provided an excellent example of this unexpected behavior. This series also played a significant role in Weierstrass’ counter-example of a continuous nowhere differentiable function (see Section 6).

## 5. Further criticism of Riemann’s methods

Apparently, Riemann’s theory of complex functions seems to have been the background of Weierstrass’ work and lectures. Evidence of this is provided by his (unpublished) correspondence with his former student H. A. Schwarz from 1867 up to 1893. One of the first topics they discussed was Riemann mapping theorem. In his thesis Riemann had claimed that “two given simply connected plane surfaces can always be mapped onto one another in such a way that each point of the one corresponds to a unique point of the other in a continuous way and the correspondence is conformal; moreover, the correspondence between an arbitrary interior point of the one and the other may be given arbitrarily, but when this is done the correspondence is determined completely” ([7], 40). Riemann’s proof of



the mapping theorem rested on a suitable application of Dirichlet’s principle. Because of his criticism of this principle, in Weierstrass’ view the Riemann mapping theorem remained a still-open question, worthy of a rigorous answer. Following Weierstrass’ suggestion, Schwarz tackled this question after his student days and succeeded in establishing the theorem in particular cases, without resorting to the questionable principle. In a number of papers he gave the solution of the problem of the conformal mapping of an ellipse - or, more generally, of a plane, simply connected figure, with boundaries given by pieces of analytic curves which meet to form non-zero angles - onto the unit disk, by using suitable devices as the lemma and the reflection principle, both named after him ([8], vol. 2, 65-132).

In 1870 Schwarz discovered his alternating method. “With this method - he stated by presenting it in a lecture - all the theorems which Riemann has tried to prove in his papers by means of the Dirichlet principle, can be proved rigorously” ([8], vol. 2, 133). He submitted to Weierstrass an extended version of the paper, and in a letter of July 11, 1870 Schwarz asked him whether he had “objections to raise”. Apparently, Weierstrass’ answer has been lost. It is quite significant, however, that three days later, on July 14, 1870 Weierstrass presented to the Berlin Academy his celebrated counterexample to the Dirichlet principle ([10], vol. 2, 49-54), and then submitted Schwarz’s 1870 paper for publication in the *Monatshefte* of the Academy. Two years later, in a letter of June 20, 1872 Schwarz called Weierstrass’ attention to the still widespread idea that a continuous function always is differentiable. As the French mathematician Joseph Bertrand had made this claim in the opening pages of his *Traité*, Schwarz ironically wondered about asking Bertrand to prove that

$$f(x) = \sum_{n \geq 1} \frac{\sin n^2 x}{n^2} \quad (5.1)$$

has a derivative. One month later, on July 18, 1872 Weierstrass presented the Academy with his celebrated example of a continuous, nowhere differentiable function

$$f(x) = \sum_{n \geq 0} b^n \cos a^n x \pi \quad (5.2)$$

where  $a$  is an odd integer,  $0 < b < 1$ , and  $ab > 1 + \frac{3}{2}\pi$ . According to Riemann’s students, Weierstrass remarked, the very same function 5.1) mentioned by Schwarz had been presented by Riemann in 1861 or perhaps even earlier in his lectures as an example of continuous nowhere differentiable function. “Unfortunately Riemann’s proof has not been published”, Weierstrass added, and “it is somewhat difficult to prove” that 5.1) has this property, he concluded before producing his own example ([10], vol. 2, 71-74).

Only by the end of 1874 was Weierstrass able to overcome a major difficulty which for a long time had prevented him from building a satisfactory theory of single-valued functions of one variable. This was the proof of the representation theorem of a single-valued function as a quotient of two convergent power series. As he wrote on the same day (December 16, 1874) to both Schwarz and S. Kovalevskaya, this was related to the following question: given an infinite sequence of constants

$\{a_n\}$  with  $\lim |a_n| = \infty$  does there always exist an entire, transcendental function  $G(x)$  which vanishes at  $\{a_n\}$  and only there? He had been able to find a positive answer to it by expressing  $G(x)$  as the product  $\prod_{n \geq 1} E(x, n)$  of "prime functions"

$$E(x, 0) = 1 + x,$$

...

$$E(x, n) = (1 + x) \exp\left(\frac{x}{1} + \frac{x^2}{2} + \cdots + \frac{x^n}{n}\right)$$

which he introduced there for the first time. The "until now only conjectured" representation theorem followed easily. This theorem constituted the core of Weierstrass's 1876 paper on the "systematic foundations" of the theory of analytic functions of one variable ([10], vol. 2, 77-124). In spite of his efforts, however, he was not able to extend his representation theorem to single-valued functions of several variables. "This is regarded as unproved in my theory of Abelian functions" Weierstrass admitted in his letter to Kovalevskaya. (For 2 variables this was done by Poincaré in 1883 and later extended by Cousin in 1895 following different methods from Weierstrass'). Four days later Weierstrass wrote to Schwarz stating that Riemann's (and Dirichlet's) proof of Cauchy integral theorem by means of a double integration process was in his opinion not a "completely methodical" one. On the contrary, a rigorous proof could be obtained by assuming the fundamental concept of analytic element (and its analytic continuation) and by resorting to Poisson integral for the disk, as Schwarz himself had shown in his paper on the integration of the Laplace equation. Criticism of Riemann's ideas and methods were also occasionally expressed by Weierstrass in his letters to Kovalevskaya [6]. On August 20, 1873 he was pleased to quote an excerpt from a letter of Richelot to himself "in which a decisive preference was expressed for the route chosen by Weierstrass in the theory of Abelian functions as opposed to Riemann's and Clebsch's". On January 12, 1875 Weierstrass announced to Kovalevskaya his intention of presenting the essentials of his approach to Abelian functions in a series of letters to Richelot where he hoped "to point out the uniqueness of my method without hesitation and to get into a criticism of Riemann and Clebsch".

Weierstrass openly stated his criticism of Riemann's methods in a often-quoted "confession of faith" he produced to Schwarz on October 3, 1875: "The more I think about the principles of function theory - and I do it incessantly - the more I am convinced that this must be built on the basis of algebraic truths, and that it is consequently not correct when the 'transcendental', to express myself briefly, is taken as the basis of simple and fundamental algebraic propositions. This view seems so attractive at first sight, in that through it Riemann was able to discover so many of the important properties of algebraic functions". Of course, Weierstrass continued, it was not a matter of methods of discovery. It was "only a matter of systematic foundations" ([10], vol. 2, 235). It is worth remarking that Weierstrass added he had been "especially strengthened [in his belief] by his continuing study of the theory of analytic functions of several variable".

## 6. Weierstrass' last papers

After Mittag-Leffler, Poincaré and Picard had deeply extended the results of

his 1876 paper following “another way” different from his own, Weierstrass felt it necessary to explain his approach to complex function theory and to compare it with those of Cauchy and Riemann. He did this in a lecture that he delivered at the Berlin Mathematical Seminar on May 28, 1884 [11]. Even though “much can be done more easily by means of Cauchy’s theorem”, Weierstrass admitted, he strongly maintained that the general concept of a single-valued analytic function had to be based on simple, arithmetical operations. His discovery of both continuous nowhere differentiable functions and series having natural boundaries strengthened him in this view. “All difficulties vanish”, he stated, “when one takes an arbitrary power series as the foundation of an analytic function” ([11], 3).

Having summarized the main features of his own theory, including in particular the method of analytic continuation, he advanced his criticism of Riemann’s general definition of a complex function (*see* Section 2). This was based on the existence of first-order partial derivatives of functions of two real variables, whereas “in the current state of knowledge” the class of functions having this property could not be precisely delimited. Moreover, the existence of partial derivatives required an increasing number of assumptions when passing from one to several complex variables. On the contrary, Weierstrass concluded, his own theory could “easily” be extended to functions of several variables.

A major flaw in Riemann’s concept of a complex function had been discovered and published by Weierstrass in 1880. The main theorem of his paper stated that a series of rational functions, converging uniformly inside a disconnected domain may represent different analytic functions on disjoint regions of the domain ([10], vol. 2, 221). Thus, Weierstrass commented, “the concept of a monogenic function of a complex variable does not coincide completely with the concept of dependence expressed by (arithmetic) operations on quantities”, and in a footnote he pointed out that “the contrary statement had been made by Riemann” in his thesis. Before proving his theorem Weierstrass discussed an example he had expounded in his lectures “for many years”. By combining the theory of linear transformations of elliptic  $\theta$ -functions with the properties of the lacunary series 4.1), Weierstrass was able to prove that the series

$$F(x) = \sum_{n \geq 0} \frac{1}{x^n + x^{-n}} \quad (6.1)$$

is convergent for  $|x| < 1$ , and  $|x| > 1$ , but “in each region of its domain of convergence it represents a function which cannot be continued outside the boundary of the region” ([10], vol. 2, 211). (It is worth noting that  $1 + 4F(x) = \theta_0^2(x)$ ).

This remark allowed Weierstrass to clarify an essential point of function theory, which deeply related the problem of the analytical continuation of a complex function to the existence of real, continuous nowhere differentiable functions. In order to explain this relation Weierstrass considered the series  $\sum_{n \geq 0} b^n x^{a^n}$  which is absolutely and uniformly convergent in the compact disk  $|x| \leq 1$ , when  $a$  is an odd integer,  $0 < b < 1$ . By a suitable use of his example of a continuous nowhere differentiable function 5.2), he concluded that under the additional condition  $ab > 1 + \frac{3}{2}\pi$  the circle  $|x| = 1$  reveals to be the natural boundary of the series. Contrary to his

habit, in 1886 Weierstrass reprinted this paper in a volume which collected some of his last articles, including a seminal paper where he stated his celebrated “preparation theorem” together with other theorems on single-valued functions of several variables that he used to expound in his lectures on Abelian functions [9].

## 7. Conclusion

From the 1840s to the end of his life Weierstrass continued to study the theory of Abelian functions, devoting an incredible amount of work to the topic. This theory was the background of many of the results he presented in his papers and lectures, or discussed in his letters to colleagues. In spite of his efforts, however, Weierstrass never succeeded in giving it the complete, rigorous treatment he was looking for. The huge fourth volume of his *Mathematische Werke* (published posthumously) collects the lectures on Abelian functions he gave in Winter semester 1875–76 and Summer semester 1876. Two thirds of it is devoted to algebraic functions and Abelian integrals, and only the remaining one third to the (general) Jacobi inversion problem. Thus, the editors of the volume, Weierstrass’ former students G. Hettner and J. Knoblauch, could aptly state in the preface that the theory of Abelian functions (in Weierstrass’ sense) “is sketched only briefly” there. It was not an irony of the history if Weierstrass failed in his pursuit of his his main mathematical goal whereas the machinery that he created to attain it in response to Riemann’s “geometric fantasies” became an essential ingredient of modern analysis. The contradiction between Weierstrass’ approach, in which all geometric insight was lacking, and Riemann’s geometric one remained effective until the early decades of the 20<sup>th</sup> century, when the theory of functions of several complex variables began to be established in modern terms.

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