An Introduction to Knot Theory

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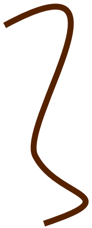
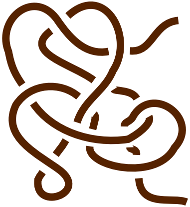
- Grab one of your shoelaces or just imagine a piece of rope.



rope

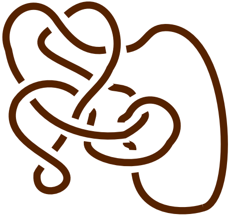
What Is A Knot?

- Take your shoelace (or your imaginary piece of rope) and make it into a "tangled-up mess".



tangled rope

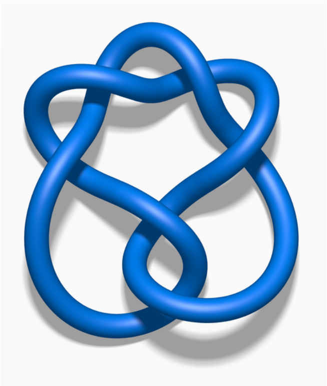
- Take the two ends of this "tangled-up mess" and glue them together, making a "tangled-up mess that's now a loop".



tangled loop of rope

What Is A Knot?

- A **knot** is a mathematical model of a “tangled up loop of rope” that lives in three dimensions.



When Are Two Knots "The Same"?

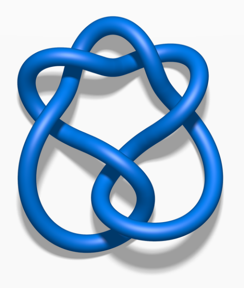
- If you take the knot and play with it, it's still the same knot.
- This means we want to allow ourselves to deform a knot but still think of it as the same knot.
- Like in the real world, we do NOT want:

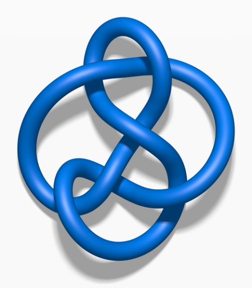
(1) to allow our knot to be cut open or broken apart.
(2) to allow one strand of the knot to pass through another strand.
(3) to allow our knot to magically shrink down to a point.
(4) to allow ourselves to create any kind of "infinite knotting".

When Are Two Knots "The Same"?

- Two knots K_1 and K_2 are **equivalent** if we can deform K_1 in three dimensions until it looks like K_2.

Challenging Question: Are the two knots depicted below equivalent or are they nonequivalent?





Quick History and Applications Break

Question: Why would anyone ever want to figure out if knotted loops are the same or different?

- In the 1800s, some scientists believed that space contained an invisible substance called “the ether”.
- In 1867, Lord Kelvin (William Thompson) proposed an early theory of the atom that modeled each atom as its own “knotted tube of ether”.
- This led physicists like Peter Tait to work on creating a “periodic table of knots”.

Quick History and Applications Break

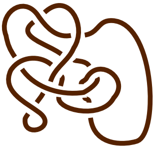
- Eventually the 1913 atomic model of Neils Bohr won out, but a new field of math was born from the earlier model.
- But don't worry science people! In the past 30 years or so, knot theory has been applied to parts of:

  (1) biology
    - Enzymes can change crossings in knotted DNA to eventually unknot it.
    - Having the DNA look simpler is needed for DNA replication and transcription.

  (2) chemistry
    - Viewing knots as made up of "sticks" allows us to use knots to study molecular structures.
    - Knotted molecules were synthesized in the 1980s.

Can We Draw Knots On Paper?

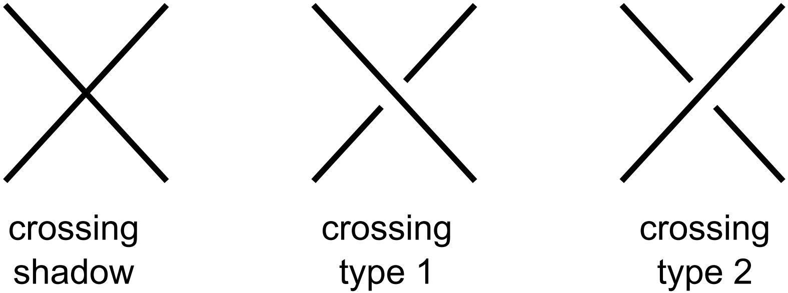
- **Issue:** A knot lives in THREE dimensions but things like paper, chalkboards, whiteboards, laptop screens, and projector screens all have TWO dimensions.
- **Big Question:** Can we actually study knots with only TWO dimensions available to draw in?
- **Satisfying Answer:** YES! (We already secretly used this fact.)



tangled loop of rope

Can We Draw Knots On Paper?

- Start with a knot in three dimensions.
- Project the knot onto a piece of paper. This is like looking at a "shadow" of the knot.
- Since we want to keep the "over" and "under" crossing information, place little gaps at each crossing.

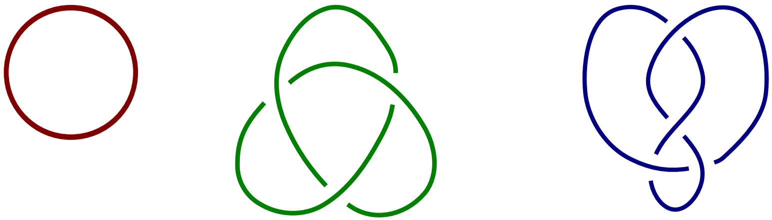


Can We Draw Knots On Paper?

Math Result:

Allowing ourselves to "tweak" the projection if needed, we can ALWAYS represent a knot K with a two-dimensional **knot diagram** $D(K)$.

Diagrams of the Three Simplest Knots:



unknot trefoil knot figure-eight knot

When Are Two Knots “The Same”? (Continued)

Issue: Knot equivalence in three dimensions is hard to visualize!

Another Big Question: Can we study knots and equivalence of knots (a THREE-dimensional problem) by instead studying knot diagrams and equivalence of knot diagrams (a TWO-dimensional problem)?

Another Satisfying Answer: YES!

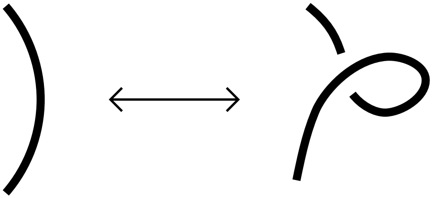
When Are Two Knots "The Same"? (Continued)

Math Result (due to Reidermeister):

Saying that the knots K_1 and K_2 are equivalent is the same thing as saying that the knot diagrams $D(K_1)$ and $D(K_2)$ are related by a finite sequence of moves called **Reidermeister moves**.

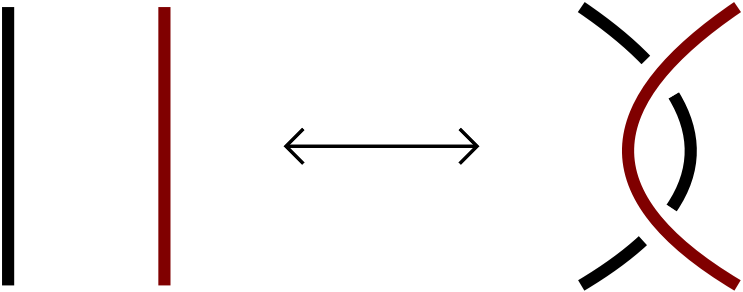
Question: What are these **Reidermeister moves**?

First Reidermeister Move: Add a little loop to the knot diagram (or the reverse process).



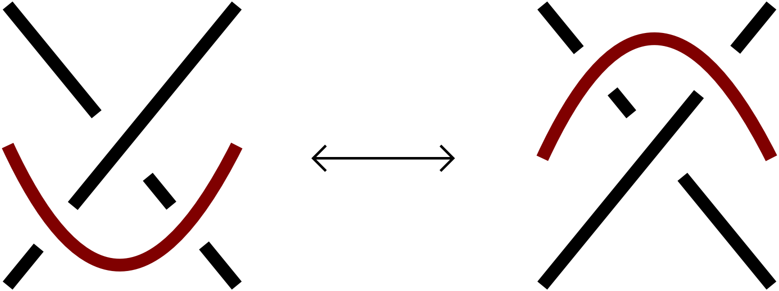
When Are Two Knot Diagrams "The Same"?

Second Reidermeister Move: Slide one strand of a knot diagram over another strand (or the reverse process).



When Are Two Knot Diagrams "The Same"?

Third Reidermeister Move: Slide a strand of the knot diagram over a crossing.



Knot Invariants

- One of the coolest strategies to study knots is to use **knot invariants**.
- A **knot invariant** is something that doesn't change when we look at equivalent knots.
- Knot invariants are useful because they can be used to try to tell knots apart (to make a "periodic table of knots").

Reidermeister's Result:

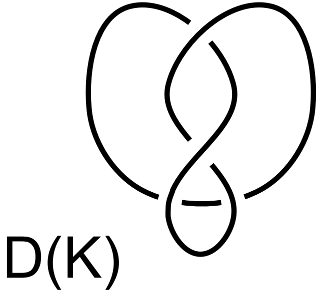
Saying that the knots K_1 and K_2 are equivalent is the same thing as saying that the knot diagrams $D(K_1)$ and $D(K_2)$ are related by a finite sequence Reidermeister moves.

Punchline: To show that something is a knot invariant, all we have to do is show that this thing doesn't change when we apply each of the three Reidermeister moves.

- Let's talk about a colorful knot invariant called **tricolorability**.

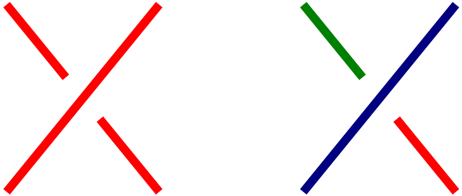
Tricolorability

- Start with a diagram $D(K)$ of a knot K.



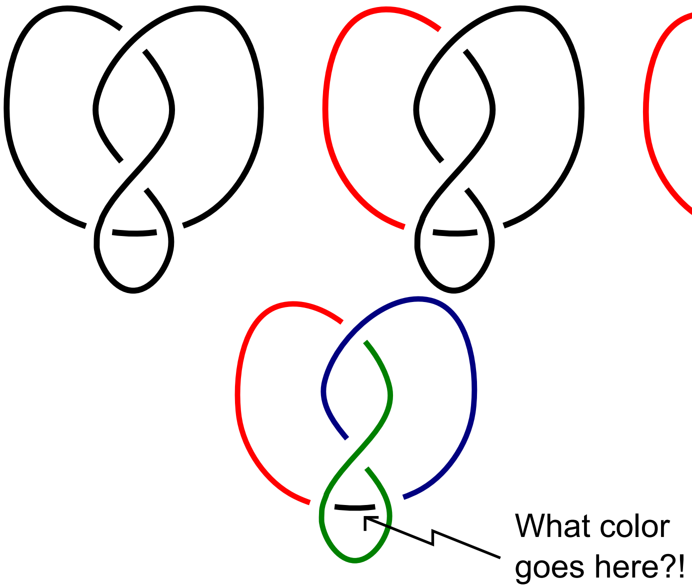
Tricolorability Rules:

- Color each strand of $D(K)$ using one of three possible colors so that:
  (1) all three colors get used
  (2) either **all the same color** OR **all three colors** meet at each crossing



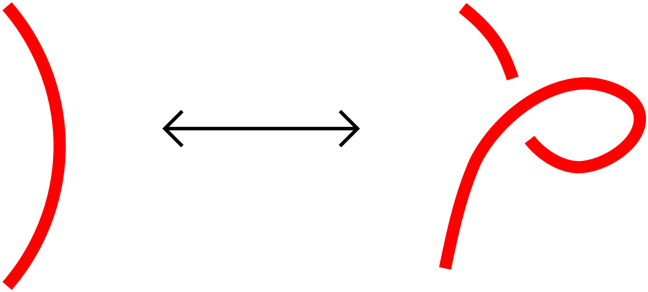
- A knot K is called **tricolorable** if any diagram $D(K)$ of K can be colored in a way that satisfies the Tricolorability Rules.

Tricolorability

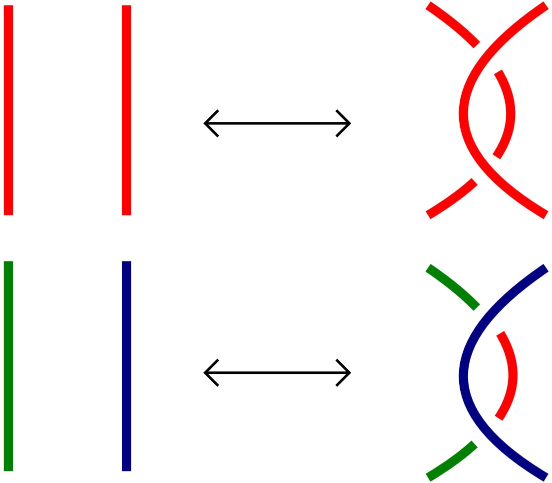


Goal: Show that tricolorability is actually a knot invariant.

Invariance Under the First Reidermeister Move:

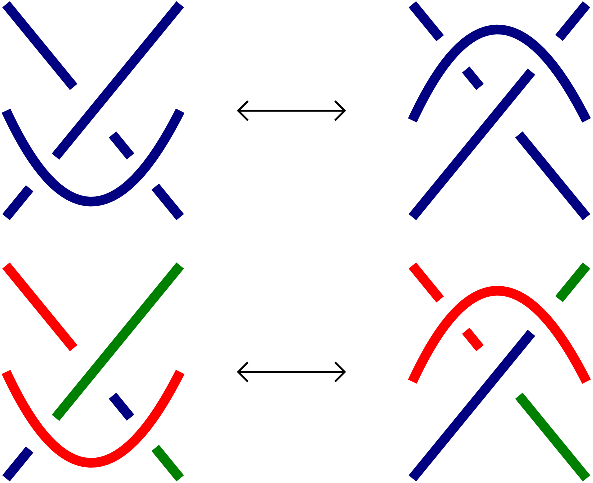


Invariance Under the Second Reidermeister Move:

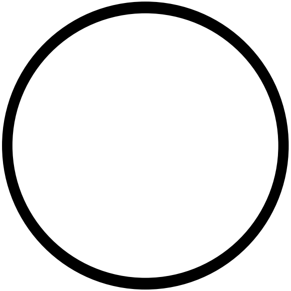


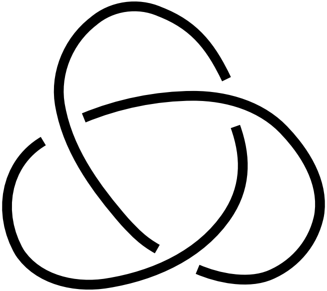
Tricolorability

Invariance Under the Third Reidermeister Move:

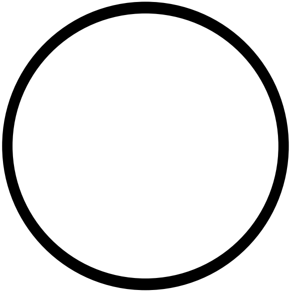


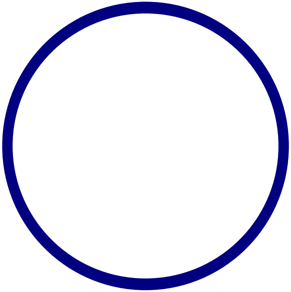
New Goal: Show that the first two knots in our "periodic table of knots" are actually different knots.





Notice: The unknot diagram DOES NOT satisfy the Tricolorability Rules because NOT all three colors are used.

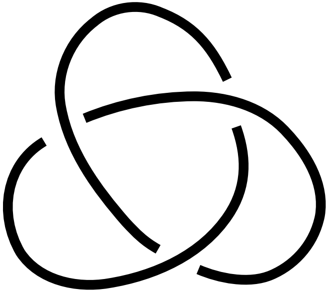


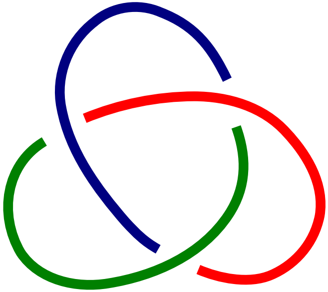


Conclusion: The unknot IS NOT tricolorable.

Tricolorability

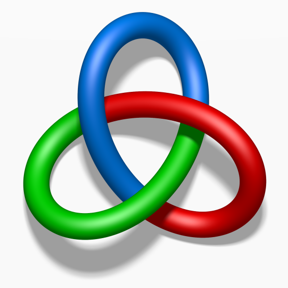
Notice: The trefoil knot diagram DOES satisfy the Tricolorability Rules.





Conclusion: The trefoil knot IS tricolorable.

Tricolorability



Conclusion: Since the unknot IS NOT tricolorable and the trefoil knot IS tricolorable, then these knots must be different knots!

Thank You!

Questions?

References

- *The Knot Book* by Colin Adams
- *Why Knot: An Introduction to the Mathematical Theory of Knots* by Colin Adams