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# AN OVERVIEW OF THE APPLICATIONS OF MULTISETS[1]

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**Abstract.** This paper presents a systemization of representation of multisets and basic operations under multisets, and an overview of the applications of multisets in mathematics, computer science and related areas.

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## 1. Introduction

A multiset (mset, for short) is an unordered collection of objects (called the elements) in which, unlike a standard (Cantorian) set, elements are allowed to repeat. In other words, an mset is a set to which elements may belong more than once, and hence it is a non-Cantorian set. In this paper, we endeavour to present an overview of basics of multiset and applications.

The term multiset, as Knuth ([46], p. 36) notes, was first suggested by N.G.de Bruijn in a private communication to him. Owing to its aptness, it has replaced a variety of terms, viz. list, heap, bunch, bag, sample, weighted set, occurrence set, and fireset (finitely repeated element set) used in different contexts but conveying synonimity with mset.

As mentioned earlier, elements are allowed to repeat in an mset (finitely in most of the known application areas, albeit in a theoretical development infinite multiplicities of elements are also dealt with (see [11], [37], [51], [72] and [30], in particular).

The number of copies (([17], P. 5) prefers to call it 'multiples') of an element appearing in an mset is called its multiplicity. Moreover, multiple occurrences of an element in an mset are treated without preference (perhaps to retain the force of classical concept of identity). We mention [64] for an earliest extensive treatment of indistinguishability of repeated elements without any preference, and [85] for an alternative treatment.

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The number of distinct elements in an mset $M$ (which need not be finite) and their multiplicities jointly determine its cardinality, denoted by $C(M)$. In other words, the cardinality of an mset is the sum of multiplicities of all its elements. An mset $M$ is called finite if the number of distinct elements in $M$ and their multiplicities are both finite, it is infinite otherwise. Thus, an mset $M$ is infinite if either the number of elements in $M$ is infinite or the multiplicity of one or more of its elements is infinite, i.e. $C(M) \geq \aleph_0$.

The root or support or carrier of an mset $M$, denoted by $M^*$, is defined as follows:

$M^* = \{x \in M \mid M(x) > 0\}$.

The elements of the root set of an mset are called the generators of that mset.

A considerable amount of efforts have also gone into the study of msets with negative multiplicities (see [9], [36], [74], [72], [30], [84], [85], in particular).

## 2. Basics of Multiset

### 2.1. Representations of Multisets

#### 2.1.1. Multiplicative form

Following Meyer and McRobbie [57], the use of square brackets to represent an mset has become almost standard. Thus, an mset containing one occurrence of $a$, two occurrences of $b$, and three occurrences of $c$ is notationally written as $[[a, b, b, c, c, c]]$ or $[a, b, b, c, c, c]$ or $[a, b, c]_{1,2,3}$ or $[a^1, b^2, c^3]$ or $[a1, b2, c3]$, depending on one's taste and convenience.

#### 2.1.2. Linear form

Wildberger [85] puts forward a linear notation for multisets, which seems quite innovative, especially when negative multiplicities (integral as well as rational) are to be dealt with. For example, the mset $M = [a, b, c]_{1,2,3}$ can be written as

$M = [a] + 2[b] + 3[c]$.

Similarly, a rational mset can be represented, for example,

$N = \frac{2}{3}[5] - \frac{1}{2}[1\_8]$.

In order to accommodate negative multiplicities round brackets are used: $(a)$ in an mset stands for negative of $a$; for example,

$[2, 4, (5), (5), 4] = [2] + 2[4] - 2[5]$.

In the same place, the distinction between the terms 'element' and 'object' occurring in an mset is made explicit as fallows:

Each individual occurrence of an object $x$ in an mset $A$ is called an element of $A$. Thus, in the linear notation of $M$ above; $b$, for example, is an object appearing twice, and every occurrence of $b$ is an element of $M$. It follows that the distinct elements of an mset are the objects. An object is an element if its multiplicity is unity.

Further, the following notations used in ([85], pp. 5–6) to represent data structures of set, ordered set, multiset and list, are quite instructive:

A collection containing $x, y, z, \dots$ is denoted by $\{xyz\dots\}$ or $\{x\_y\_z\_\dots\}$ if it is a set; $\{x, y,\ z\dots\}$ if it is an ordered set; $[xyz\dots.] = [x\_y\_z\_...]$ if it is a multiset; and

$[x, y, z, \dots]$ if it is a list.

Note that a list is an ordered sequence of elements with repetitions allowed, whereas an mset is a sequence with its ordering stripped off.

### 2.1.3. Multiset as a Sequence

A multiset can also be represented as a sequence in which the multiplicity of an element equals the number of times the element occurs in the sequence, which is exactly Dedekind's 'frequency-number'. The idea is to construct an mset as a sequence (a function with domain $\aleph$, the set of natural numbers) and ignore the ordering of its elements, which can be done by taking all permutations of the domain of the sequence.

### 2.1.4. Multiset as a Family of Sets

A multiset can also be represented as a family of sets, which is altogether a generalization of the idea of a sequence described above. Thus, the family of sets $F = \{F_i\}, i \in I$, where $F_i = F_j$, if $i = j$, which identifies a repeated element, represents an mset. Clearly, such a family $F$ is a function: $I \rightarrow \{F_i |\ i \in I\}$, which in turn, is a sequence if $I = \aleph$.

### 2.1.5. Multiset as a Numeric-valued function

Representation of an mset as a numeric-valued or cardinal-valued function abounds, especially in the application areas. Formally, an mset is just a mapping from some ground or generic or universal set into some set of numbers. For example, an mset

$A = [x, y, z]_{1,2,3}$ is a mapping from a ground set $S$ to $\aleph$, the set of natural numbers with zero, defined by

$$\alpha(t) = \begin{cases} 1, & \text{if} \qquad t = x \\ 2, & \text{if} \qquad t = y \\ 3, & \text{if} \qquad t = z \\ 0, & \text{for} \quad \text{all the remaining } t \in S. \end{cases}$$

In general terms, for a given ground set $S$ and a numeric set $T$, we call a mapping

$\alpha : S \longrightarrow T$,

$$\begin{cases} \text{a set, if } T = \{0, 1\}\,; \\ \text{a multiset, if } T = \aleph, \text{the set of natural numbers;} \\ \text{a signed multiset (or, hybrid/shadow set) if } T = Z, \text{ the set of integers;} \\ \text{a fuzzy (or hazy) set if } T = [0, 1] \subseteq R, \text{ a two-valued Boolean algebra.} \end{cases}$$

#### 2.1.6. Multiset as a generalized characteristic function

Similar to the representation of a set by its characteristic function (function whose range is $\{0, 1\}$), a multiset or hybrid set is determined by its generalized characteristic function (whose range is the set of integers, positive, negative or zero), see [84] for details.

### 2.2. Operations under Mset

The monograph [46] can be considered as the earliest reference describing intuitively properties of msets in a sufficient detail. During the recent years, a good number of papers ([51], [36] , [37], [8], [85], and others) have appeared. We endeavour to present an overview of various approaches in this regard. We will adhere to function-approach and use $\mathrm{Dom}(f)$, $\mathrm{Ran}(f)$ to denote the domain and range respectively of a given function f.

**Definition 1.** ***Multiset***

Let $D = \{x_1, x_2, \ldots, x_j, \ldots\}$ be a set. An mset $A$ over $D$ is a cardinal-valued function, i.e. $A : D \to \aleph = \{0, 1, 2, ...\}$ such that for $x \in \mathrm{Dom}(A)$ implies $A(x)$ is a cardinal and $A(x) = m_A(x) > 0$, where $m_A(x)$ denotes the number of times an object $x$ occurs in $A$, i.e. a counting function of $A$. The set $D$ is called the ground or generic set of the class of all msets containing objects from $D$.

An mset $A$ can also be represented by the set of pairs as follows:

$A = \{\langle m_A(x_1),\ x_1\rangle, ..., \langle m_A(x_j),\ x_j\rangle, \ldots\}$

or, $A = \{m_A(x_1).x_1, ..., m_A(x_j).x_j, ...\}$.

Relatedly, an mset is called 'regular' or 'constant' if all its objects occur with the same multiplicity. Also, an mset is called 'simple' if all its objects are the same, for example, $[x]_3$ is a simple mset containing $x$ as its only object (see [8]). Clearly, the root set of every simple mset contains a single object.

**Definition 2.** ***Dressed epsilon symbol,*** $\in_+$

The symbol $\in_+$ was first introduced by Singh and Singh [80]. For any object $x$ occurring as an element of an mset $A$, i.e. $m_A(x) > 0$, we write $x \in_+ A$, where $\in_+$ (dressed epsilon is a binary predicate intended to mean 'belongs to at least once', as $\in$ is 'belongs to only once' in the case of sets. Thus, $m_A(x) = 0$ implies $x \notin A$, and $x \in_+^k A$ implies '$x$ belongs to $A$ at least $k$ times', however $x \in^k A$ means '$x$ belongs $k$ times to $A$'. The mset for any ground set $D$ is called empty, denoted by $\varnothing$ or [ ], if $m_\varnothing(x) = 0$ for all $x \in D$. Further, in order to make our presentation concise, we shall follow some terminologies introduced in ([37], pp. 212 – 213): ('A(x) denotes the number of copies of x, including x itself, belonging to Dom (A)'), which is exactly the Dedekind's frequency number.

**Definition 3.** ***Multisubsets (or msubsets, for short)***

Let $A$ and $B$ be two msets, $A$ being an msubset or a submultiset of $B$, written as $A \subseteq B$ or $B \supseteq A$, if $m_A(x) \le m_B(x)$ for all $x \in D$. Also, if $A \subseteq B$

and $A \neq B$, then $A$ is called a proper submset of $B$. An mset is called the parent in relation to its msubsets.

It follows from the definition of multisets that $A = B$ if and only if for $x \in D$, $m_A(x) = m_B(x)$. Also, $A = B \Longrightarrow A^* = B^*$, but the converse need not hold. It is easy to see that $\subseteq$ is antisymmetric, i.e. $A \subseteq B$ and $B \subseteq A \Longrightarrow A = B$, and it is a partial ordering on the class of msets defined on a given generic domain. Clearly, $\varnothing$ is a submset of every mset. Note that the terms 'element' and 'object' are being distinguished throughout, and coincide if a generic set is in consideration. We wish to emphasize that introduction of $\in_+$ greatly enhance the language of msets. For example, $A \subseteq B$ stands for

$\forall z \forall k (z \in^k A \longrightarrow z \in_+^k B)$.

Relatedly, a 'whole' msubset of a given mset contains all multiplicities of common elements; while a 'full' msubset contains all objects of the parent mset, and accordingly, every mset contains a unique full msubset, its root set. Clearly, for any two msets $A$ and $B$, if $A \subseteq B$ and $Dom(A) = Dom(B)$, then $A$ is a full msubset of $B$.

**Definition 4.** ***Similar msets***

Two msets $A$ and $B$ are said to be 'cognate' or 'similar' if $\forall x(x \in A \Longleftrightarrow x \in B)$, where $x$ is an object. Thus, similar msets have equal root sets but need not be equal themselves.

**Definition 5.** ***Ordered pair of two mset terms***

Ordered pair of two mset terms $u$ and $v$, denoted by $[u, v]$, can be defined as follows:

$[u, v] = \{u, v\}$ if $u \neq v$, and $[u, v] = \{[u]_2\}$ if $u = v$.

Here, $[u]$ is written as $\{u\}$, and $\langle u, v\rangle$ is actually the ordered pair set, where Set $(u)$ stands for $u = \varnothing \vee \forall x \forall n\ (x \in^n u \Longrightarrow n = 1)$, though $x$ itself may be an mset term, (see [8] pp. 42 – 44, for details).

**Definition 6.** ***Power multiset***

In Cantorian spirit, the power multiset of a given mset $A$, denoted by $\tilde{\wp}(A)$ to distinguish it from the symbol $\wp(A)$ used for power set of $A$, is the multiset of all submultisets of $A$. For example, let $A = [x, y]_{2,1} = [x, x, y]$. Then,

$\tilde{\wp}(A) = [\phi, \{x\}, \{x\}, [x]_2, \{y\}, \{x, y\}, \{x, y\}, [x, y]_{2,1}]$.

In this sense, $C(\tilde{\wp}(A)) = 2^{C(A)}$, for any mset $A$. However, as has been voiced by many researchers in the area of msets and their applications (see [37], p. 213 and [8], p. 45, in particular), there is no 'good' reason for admitting repeated elements into a power multiset. Hence, a power multiset needs to be called a power set only and denoted by $\wp(A)$. Accordingly, for $A = [x,\ y]_{2,1}$; $\wp(A) = [\phi, \{x\}, [x]_2, \{y\}, \{x, y\}, [x, y]_{2,1}]$ and hence $C(\wp(A)) < 2^{C(A)}$ which implies that Cantor's power set theorem: $C(A) < C(\wp(A))$ fails. However, for finite msets, Cantor's theorem holds for power mset (see [8], p. 45, for related inherent difficulties if the mset in consideration is infinite).

**Definition 7.** ***Union (∪), Intersection (∩) and addition or sum or merge ⊎ or (+)***

Let $A$ and $B$ be two msets over a given domain set $D$.

**1.** $A \cup B$ is the mset defined by $m_{A\cup B}(x) = m_A(x) \cup m_B(x) =$ maximum $(m_A(x),\, m_B(x))$, being the union of two numbers. That is, an object $z$ occurring $a$ times in $A$ and $b$ times in $B$, occurs maximum $(a, b)$ times in $A \cup B$, if such a maximum exists; otherwise the minimum of $(a, b)$ is taken, which always exists.

It follows that for any given mset $x$ there exists an mset $y$ which contains elements of elements of $x$, where the multiplicity of an element $z$ in $y$ is the maximum multiplicity of $z$ as an element of elements of $x$ along with the above stipulation on the existence of such a maximum. We denote this fact by $y = \cup x$

Clearly, $Dom(\cup x) = \cup\{Dom(A),\ A \in x\}$ and that the multipliticity of $z$ in $y$ is the maximum of its multiplicities as an element of elements of $x$, if it exists; otherwise, the minimum is taken.

For example, if $A = [2\ 3\ 4\ 4], B = [1\ 4\ 3\ 3]$ then $A \cup B = [1\ 2\ 3\ 3\ 4\ 4]$.

Also, it follows that for a finite mset $x$, the maximum multiplicity of elements of elements of $x$ always exists. However, for certain infinite sets like $x = \{\{y\}, [y]_2, [y]_3 \ldots\}$, the maximum multiplicity of elements of elements of $x$ does not exist, and hence $\cup x = \{y\}$. It is obvious by definition of the union that multiplicity of any $y \in x \neq \varnothing$ is irrelevant to $\cup x$, and hence $\cup x = \cup x^*$ ( see [8], pp. 48 – 49, for details).

**2.** $A \cap B$ is the mset defined by $m_{.A\cap B}(x) = m_A(x) \cap m_B(x)$ $=$ minimum$(m_A(x),\, m_B(x))$, being the intersection of two numbers.

That is, an object $x$ occurring $a$ times in $A$ and $b$ times in $B$, occurs minimum $(a, b)$ times in $A \cap B$, which always exists.

In general, for a given mset $x$, $Dom(\cap x) = \cap\{Dom(A) : A \in x\}$ and $z \in \cap x$ implies that the multiplicity of $z$ is the minimum of its multiplicities as element of elements of $x$.

For example, if $A = [3\ 3\ 3\ 4\ 4]$, $B = [1\ 4\ 3\ 3]$, then $A \cap B = [3\ 3\ 4]$.

Note that for any mset $x$, we have $\cap x \subseteq \cup x$.

**3.** $A \uplus B$ is the mset defined by $m_{A \uplus B}(x) = m_A(x) + m_B(x)$, direct sum of two numbers.

That is, an object $x$ occurring $a$ times in $A$ and $b$ times in $B$, occurs $a + b$ times in $A \uplus B$.

For example, if $A = [1, 1, 2, 2, 4, 4, 4]$, $B = [1, 2, 3, 3]$, then $A \uplus B = [1, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4]$.

Clearly, $C(A \uplus B) = C(A \cup B) + C(A \cap B)$.

Note that if $x$ be an infinite mset, then the multiplicity of some mset $z \in \uplus x$ may not be finite. In that case, the multiplicity of $z$ in $\cup x$ is used. For example,

if $x = \{\{z\}, [z]_2, [z]_3, ...\}$ then $\uplus x = \cup x = \{z\}$ (see ([8], p. 51, for details).

**4. Some Properties holding for mset operations**

(see [46] p. 636; [8], p. 53, in particular).

(i) Commutativity: $A \uplus B = B \uplus A$, $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(ii) Associativity: $A \uplus (B \uplus C) = (A \uplus B) \uplus C$, $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

(iii) Idempotency: $A \cup A = A$, $A \cap A = A$, but $A \uplus A \neq A$.

In fact, as it has been suggested recently (see [85], p. 9, in particular), in order to obtain a linear combination of msets, $kA$ may be interpreted to denote the sum of $k$ number of $A$'s, where $k$ is a natural number.

(iv) Identity laws: $A \cup \phi = A$, $A \cap \phi = \phi$, $A \uplus \phi = A$.

(v) Distributivity: $A \uplus (B \cup C) = (A \uplus B) \cup (A \uplus C)$,
$A \uplus (B \cap C) = (A \uplus B) \cap (A \uplus C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

The proof of all these identities follows from the interpretation of $\cup$, $\cap$ and $\uplus$ of two natural numbers as maximum, minimum and (direct) sum respectively. It is easy to see that $\uplus$ is stronger than both $\cup$ and $\cap$ in the sense that neither $\cup$ nor $\cap$ distributes over $\uplus$, whereas $\uplus$ distributes over both $\cup$ and $\cap$. Also, $\cap x \subseteq \cup x \subseteq \uplus x$

It is promising to observe that multiset operations form a "realm" ([85], p. 9).

**Definition 8.** ***Difference and complementation***

Let $A$ and $B$ be two msets over $D$, and $B \subseteq A$, then $m_{A-B}(x) = m_A(x) - m_{A \cap B}(x)$, for all $x \in D$. It is sometimes called the arithmetic difference of $B$ from $A$. Note that even if $B$ is not contained in $A$, this definition holds good. It can be seen quickly that some of the consequences of the aforesaid definition are disturbing ([37], p. 214). For example, if $A = [a, b]_{4,5}; B = [a, b]_{2,3}$ then $A - B = [a\ b]_{2,2} \subseteq B$, contradicting the classical law: $(A - B) \cap B = \phi$.

In order to define the complement, we follow Petrovsky [67]:

Let $\Im = \{A_1, A_2, ...\}$ be a family of multisets composed of the elements of the generic set $D$. Then, the maximum multiset $z$ is defined by $m_z(x) = \max_{A \in \Im} m_A(x)$ for all $x \in D$ and all $A \in \Im$.

Now, the complement of an mset $A$, denoted by $\bar{A}$, is defined as fallows:

$\bar{A} = Z - A = \{m_{\bar{A}}(x).x / m_{\bar{A}}(x) = m_z(x) - m_A(x)$, for all $x \ \varepsilon\ D\}$.

It is understood that some new operations like arithmetic multiplication, raising to the arithmetic power, direct product, raising to the direct power, defined by Petrovsky, can be gainfully exploited for further research.

**Definition 9.** ***Functions between msets***

The underlying assumption in defining a function between msets has been invariably not to allow mapping of identical elements to non-identical elements and hence, it amounts to defining the function between their root sets, which is just the classical definition of a function.

The function $f : A \longrightarrow B$ is an injection iff

(i) $f : A^* \longrightarrow B^*$ is an injection, and

(ii) $\forall z(z \in A^* \Longrightarrow m_A(z) \leq m_B(f(z)))$.

The function $f : A \longrightarrow B$ is a surjection iff

(i) $f : A^* \longrightarrow B^*$ is a surjection, and

(ii) $\forall z(z \in A^* \Longrightarrow m_A(z) \geq m_B(f(z)))$.

The function $f : A \longrightarrow B$ is a bijection iff

(i) $f : A^* \longrightarrow B^*$is a bijection and

(ii) $\forall z(z \in A^* \Longrightarrow m_A(z) = m_B(f(z)))$.

For example, $f : [x]_3 \longrightarrow [x]_{10}$ is an injection, $f : [x]_5 \longrightarrow [y]_4$ is a surjection, and

$f : [x, y]_{6,2} \longrightarrow [x, y]_{3,4}$ is neither an injection nor a surjection. For various other details, see [8] and [37]. Note that some of the consequences of the above definitions are conflicting with some fundamental theorems of the classical set theory.

1. Having defined functions between msets as above, it can be proved that Cantor's theorem does not hold, there is no injection from

$A \longrightarrow \wp(A)$, see ([37]) p. 215).

2. Msets of equal cardinality need not have a bijection between them. For example, $[a, b]_{1,2}$ and $[a, b, c]$ both contain three elements, but there can be no bijection between them because the objects and their multiplicities are different in the domain and the range of any such function. In other words, there is no bijection between their root sets, viz; $f : \{a, b\} \longrightarrow \{a, b, c\}$ can not be a bijection.

3. Schröder–Bernstein theorem fails (see [37]), p. 215] and ([8], p. 47).

Let $A = [x_1, x_2, ...]_{2,4,6,\ ...}$ and $B = [y_0, y_1, y_2, ...]_{1,3,5,\ ...}$

The function $f : A^* \longrightarrow B^*$ defined as $f(x_n) = y_n$ makes $f : A \longrightarrow B$ an injection so that $A \leq B$. The function $g : B^* \longrightarrow A^*$ defined by$g(y_n) = x_{n+1}$ makes $g : B \longrightarrow A$ an injection so that $B \leq A$. But there cannot be a bijection $h : A \longrightarrow B$ since all multiplications in $A$ are even and that in $B$ are odd. Note that $\leq$ is the standard dominance relation.

**Definition 10.** ***Multiset Ordering***

It seems really surprising that the seminal work of Knuth ([45], pp. 213 – 214, 241 – 242), related to multiset orderings and their applications, has escaped the attention of most of us until quite recently. According to Knuth:

*"Multiset $\mu_1$ dominates $\mu_2$ if both $\mu_1$ and $\mu_2$ contain the same number of elements and the $K$th largest element of $\mu_1$ is greater than or equal to the $K$ th largest element of $\mu_2$ for all $K$" (p. 214).*

"If $a$ and $b$ are multisets of $m$ numbers each, we say that $a \ll b$ iff $a\overset{\Lambda}{\Lambda}b = a$

(equivalently, $a \overset{\vee}{\vee} b = b$, the largest element of $a$ is less than or equal to the smallest of $b$). Thus $a \overset{\wedge}{\wedge} b \ll a \overset{\vee}{\vee} b$ (p. 241).

*"An nth level 'cascade distribution' is a multiset..." (p. 299).*

However, [28] is the earliest reference known to introducing multiset ordering and using it for proving termination of programs and term rewriting systems. In fact, it has served as a basis for host of orderings introduced in this context. We endeavour to outline the Dershowitz-Manna multiset ordering as follows:

Let $S$ be a set equipped with a partial ordering $<$ (irreflexive and transitive relation or, equivalently, a transitive but not an equivalence relation). Let $M(S)$ be the set of all finite msets $M$ on $S$, and let $\ll$ be the associated (induced by $<$) mset ordering on $M(S)$. It is easy to see that each $M$ is an mset with a finite carrier, viz; $\{x \in S : M(x) \neq 0\}$.

**The Dershowtiz-Manna Ordering:**

$M \ll N$ if there exist two msets $X$ and $Y$ in $M(S)$ satisfying:

(i) $\{\} \neq X \subseteq N$,

(ii) $M = (N - X) + Y$,

(iii) $(\forall y \in Y)\ (\exists x \in X)\ [y < x]$.

In other words, $M \ll N$ if $M$ is obtained from $N$ by removing none or at least one element (those in $X$) from $N$, and replacing each such element $x$ by zero or any finite number of elements (those in $Y$), each of which is strictly less than (in the ordering $<$) one of the elements $x$ that have been removed. Informally, we say that $M$ is smaller than $N$ in this case. Similarly, $\gg$ on $M(S)$ with $(S, >)$ can be defined. For example, let $S = (\{0, 1, 2, \ldots\} = \aleph)$, then under the corresponding multiset ordering $\gg$ over $\aleph$, the mset [3 3 4 0] is greater than each of the following msets: [3 4], [3 2 2 1 1 1 4 0] and [3 3 3 3 2 2]. The empty set $\{\}$ is smaller than any multiset. It is also easy to observe that: $[\forall y\ y \in N \Longrightarrow \exists x\ x \in M \wedge x > y] \Longrightarrow M \gg N$.

For various ramifications of the Dershowitz-Manna Ordering, see [41] and [56], in particular.

## 3. Applications

Over the years, besides sporadic evidence of the applications of mulisets in philosophy, Logic, Linguistics and Physics, a good number of them witnessed in mathematics and computer science, which have led to the formulation of a comprehensive theory of multisets. In this section, a modest attempt is made to present a comprehensive survey of various applications of multisets which is arranged under two major headings: Mathematics (especially, combinatorial and computational aspects) and computer science, along with some overlapping results placed appropriately.

### 3.1. Applications in Mathematics

An early elaborate reference is ([46], pp. 441–466; 636-667). Some of the Knuth's findings are as follows: The prime factorization of an integer $n > 0$

is a multiset $N$ whose elements are primes, where $\pi_{P \in N} = N$. Accordingly, as every positive integer can be uniquely factored into primes, one can obtain a bijection between the positive integers and the finite msets of prime numbers. For example, if $n = 2^2.3^3.17$, the corresponding mset is $N = [2, 2, 3, 3, 3, 17]$. A simple algebra of msets is also developed. The natural correspondence between a monic polynomial over the complex numbers and the unique mset containing its roots exists. That is, every monic polynomial $f(z)$ over the complex numbers corresponds in a natural way to the mset $F$ of its roots. Hence, if $f(z)$ and $g$ $(z)$ are the polynomials corresponding to the finite msets $F$ and $G$ of complex numbers, then $F+G = f(z)g(z)$, $F \cup G = lcm\ (f(z), g(z))$, and $F \cap G = gcd$ $(f(z), g(z))$.

Zeros and poles of meromorphic functions, invariants of matrices in canonical form and invariants of finite Abelian groups do correspond to multiset representations. Generating functions and nonnegative integer coefficients correspond one-to-one with msets of nonnegative integers. Thus, if $A$ and $B$ are msets of nonnegative integers, and if $G(z) = \sum_{n \in A} z^n$ and $H(z) = \sum_{n \in B} z^n$ be generating functions corresponding to $A$ and $B$, then $G(z) + H(z)$ corresponds to $A \propto B$, etc. Also, for msets $A$ and $B$, the product of Dirichlet generating functions $g(z) = \sum_{n \in A} \frac{1}{n^z}$ and $H(z) = \sum_{n \in B} \frac{1}{n^z}$ corresponds to the mset product $AB$.

Not much is known about the history of permutations of an mset. Knuth ([45], pp. 22–34) in line with some early references cited by himself, expounds the area of permutations of msets.

Goguen ([33], pp. 513–561), in course of developing a category – theoretic foundation of fuzzy set theory, investigates properties of semiring sets, as in [30], along with various applications to msets. He also discusses usefulness of msets in the study of combinatorics and formal languages. Chapin ([19] pp. 619–634; [20], pp. 255–267), while formalizing the theory of fuzzy sets and Boolean-lattice – valued sets, seems to have mset-model in mind. The interpretation of the atomic formula $\in (x, y, z)$ as "$x$ is an element of $y$ with degree of membership at least $z$" reflects seminality, (see section 3.3, definition 3, of this paper, cf. [80] 07]). Mathematics of multisets proves a potential tool to study computing of Grobner Bases and straightening laws in polynomial rings (see [15] and [3], in particular). Brink ([17], pp. 1–13) outlines an algebra of msets and shows that it could model relevant structures to a great extent, yet falls short of DeMorgan monoid.

Anderson [2] can be considered as the first sustained development of conbinatorics of msets, in particular. The area of concentration includes the study of sizes, numbers and properties of 'chains' (collections of pairwise $\subseteq$ comparable subsets) and 'antichains' (collections of pairwise $\subseteq$ comparable subsets) in the subset and msubset lattices (see [83], [73], [5], [21, 22], [9], in particular). Martin ([56], pp. 37–54) observes that many well-founded partial orderings of the set of finite msets on a given set have appeared that played significant roles in the study of invariant theory (see [62] and [31] for early references), ring theory (see [3], and [68] for some recent developments) and the theory of partitions (see[5],

[54], [7] and [11] for various recent developments). It is also mentioned (p. 37) that msets can be gainfully exploited in the study of measure theory which, however, would require tools of functional analysis, especially when the msets considered are infinite.

Singh and Singh [79] point out a fundamental problem. It is observed that the best known efficient formula ${}^nH_r = {}^{r+n-l}\ C_r$, designed to compute the number of combinations of $n$ distinct objects taken $r$ at a time when objects may occur with repetitions, includes terms of the type $[a_i]_n$, for each $i$, that are also msubsets of an mset in which each distinct element occurs unlimited number of times (hence, help determining the cardinality of the power set of an mset), turns out to be unworkable and computationally inefficient even if adjusted to be applicable, if the mset in consideration is finite or infinite, but with finite multiplicities. For example, for a 6–element mset $\alpha = [x, y, z]_{1,2,3}$ the number of all combinations of size $-6$ (repetitions allowed and consequently $[x]_6$ – like combinations included which is not an msubset of $\alpha$) is ${}^3H_6 = {}^8\ C_6 = 28$ whereas the number of $6-$msubsets of $\alpha$ is only one and that is $\alpha$ itself. Also, besides discussing computational inefficiency of Inclusion–Exclusion principle, the paper puts forward a reasonably efficient and workable formula.

Petrovsky [66] outlines some very innovative mathematical applications of msets. An axiomatic foundation for metrization of multiset spaces is developed.

Poplin [68] develops a multiset algebra (Chapter 8, pp. 58–129) in his Ph.D. thesis. The thesis, titled "The semiring of multisets", explicates that msets provide a connection between eigen values/eigen vectors equations for the Max-Plus and the nonnegative real number systems. It is shown that Max-Plus, Max-Times and the nonnegative real numbers can be viewed as a special case of msets. The guiding factors for undertaking research in the area of mathematics abound. For example, the $(Max, +)$ algebra has been extensively used in discrete event systems, transportation networks, parallel computations, project management, machine scheduling, to name a few (see [68], pp. 1–2).

The interest in multisets and subsets of commutative monoids has increased in the recent years (see [24] for various deliberations).

Hegarty and Larson ([38], pp. 1–25) study permutations $\pi$ of the natural numbers for which the numbers $\pi(n) - n$ belongs to a given (multi)subset $M$ of $Z$ (the set of integers), for all $n \in S$ (a given subset of the set of natural numbers).

### 3.2. Applications in Computer Science

References [45] and [46] are the early known references to the applications of msets in computer science. Knuth notes:

*The terminal string of a non-circular context free grammar form an mset that is a set if and only if the grammar is unambiguous ([45], p. 636).*

He introduces multisets into algorithms that compute values of $x^n$ where $x$ is a real quantity and $n$ is a large positive integer ([46], pp. 441-466). Multisets and permutations of multisets are applied in a variety of *search* and *sort* procedures ([45], p. 717 for various page numbers).

Eilenberg ([30], chapter *vi*) describes a general theory of msets and applies it to automata. Various semiring structures are exploited and also, extensions to the cases of msets and field structures are studied. Shoesmith and Smiley ([78], pp. 66–69, 113–114, 164, 211, 224), while studying multiple–conclusion logic, find the application of msets quite useful to account for "arrays" of formulae. Dersowitz and Manna [28] introduce mset ordering and ingeniously exploit it for proving termination of certain programs, which has served a basis for many alternative and equivalent orderings subsequently proposed to date. The major intent of this seminal work can be seen in the following: "... the mset ordering ... permits the use of relatively simple and intuitive termination functions in otherwise difficult termination proofs" see [28].

Huet and Oppen [39], and Jouannuad and Lescanne [41] introduced significant refinements of Dersowitz-Manna mset ordering. In [41], it is shown that the standard mset ordering is a maximal extension function. ([41], pp. 61–62) also provides an efficient implementation of Dersowitz–Manna mset ordering.

Peterson ([65], pp. 237–240) shows that the very foundation of Petri net theory, introduced by C.A. Petri in 1962, needs the use of msets (see also, [34] and [69]).

Mayer and McRobbie [57] find use of msets quite appropriate to account for how often a premise is repeated in characterizing relevance aspect of an argument. Thistlewaite, McRobbie and Meyer [82] make use of algebra of msets developed in [57] in explicating automated theorem proving for relevance logics, especially in the implementation while using the program KRIPKE. They also recognize that msets have been taken as *datatypes* in a number of programming languages [82] p. 26).

Bundy ([16], Chapter 13, pp. 225–240) exploits mset of numbers to illustrate the definition-and-conjecture-formation program of Lenat. The work of Manna and Waldinger [61] can be considered as a sustained exposition to substantiate the fact that mathematical logic plays a fundamental role in the realm of theoretical computer science. Chapter II, pp. 505–527 of [61] is solely devoted to explicating fundamentals of msets. A theory of BAG is developed by way of introducing a novel binary primitive operation, $\underset{\cdot}{O}$: if an atom $u$ has multiplicity $n \geq 0$ in a bag $x$, then $u$ has multiplicity $n+1$ in the bag $u\underset{\cdot}{O}x$.

Reisig [74] uses msets to define "relation nets". Interestingly, exploiting mathematical intent of the definition of msets as generalized (integer-valued) characteristic functions given in [84], the novel idea of "negative multiplicity" in an mset is introduced ([74], p. 126; see [9] for a detailed exposition). Reisig also outlines the concept of "multirelations" – an mset whose domain is the Cartesian product of a set of sorts ([74], pp. 126–131).

Yager ([86], pp. 23–27; [87], pp. 441–446), after developing an algebra of msets, introduces the notion of "fuzzy bag" as follows: " ... to each element $x$ in a fuzzy bag $A$ is associated a multiset containing elements $\alpha$ (real number in the interval [0, $l$]) with multiplicities $n$ (non-negative integers). The number $n$ indicates the number of times the element $x$ appears with membership grade $\alpha$ in the fuzzy bag $A$" ([86], p. 33). Yager notes that Zadeh fuzzy sets are special cases of fuzzy bags ([86], p. 35).

Grzymala – Busse ([35], pp. 325–332) extends the notion of "rough" set (introduced by Pawlak to rough mset (see also, [58, 59])).

Martin [55] constructs a number of extension functions for mset orderings. Also, Martin [56] introduces various well-founded partial orderings of the set of all finite msets whose elements are taken from a given set. Martin exploits the notion of *cone* in $R^n$ ($n-$dimensional real space) and provides a systematization of the construction and classification of various well founded partial orderings underlying in proofs of program termination (see [28], [39], [41], etc.), in term rewriting systems (see [26], [42] , [63], [55], [23], etc.), and computer algebra (see [30], [15], [29] , [68], etc.).

Pratt ([69], pp. 33–71), besides giving several arguments for the use of *pomsets*, specially showing how pomsets can be used to represent *parallel processes*. He also describes how Petri nets can be modelled as pomsets.

Gischer ([32] pp. 199–224) exploits the notion of a partial string and a partial language introduced in [34] to show how pomsets can be used as a model of concurrency.

Applications of msets in Logic Programming languages is found to overcome "computational inefficiency" inherent in otherwise situation, especially in solving a sweep of real-life problems where multiple occurrences of an identical element are persistent. In fact, both the usual options, viz. "attaching different levels" or "assigning numbers" to account for multiplicity turn out to be computationally inefficient. Representation of msets in a logic programming language can be effected, for example, by introducing a binary associative and commutative function symbol $o$, which also admits a unit element. In a constraint logic programming language, constraints are built up from mset operations and relations. Kiziltan ([43], Chapter 7, pp. 165–202) has recently dealt with multiset and strict multiset ordering constraints. Rule-based multiset programming paradigm is recently exploited to study synthetic biology (see [4], [24] and [47] for details). Basically, multisets are interpreted to represent biological systems, such as molecules in a biochemical system. A host of simulators for biological systems has been developed and found useful to several fields of biology such as biochemistry, microbiology, and evolutionary biology. For example, a system could be a multiset $[A, A, B, B, C]$, where $A$, $B$ and $C$ are elements that evolve by means of the application of a set of rewriting rules (say, of the biological systems) viz.:

$\{\{A, B\} \longrightarrow \{C\}, \{C\} \longrightarrow \{A, B\}\}$. If we choose the first rule to apply to the multiset in the example, the multiset $\{A, B, C, C\}$ is obtained (see [12], for details). The interaction among elements in a multiset object space, which includes the environment, are like chemical reactions and the evolution of multiset can mimic the biological evolution leading to plausibility of DNA computing and programmable living machines (see [47], for details).

The seminal work [28] has dominated research in termination, especially in Term Rewrite Systems (TRS), during the last two decades or so. Full surveys, particularly of division orderings on structures such as strings, partitions or terms, given in [27] and [81] [52], provide a complete development of the topic with the help of Coq Proof Assistant.

As for the present, a typical method to prove termination of a particular TRS involves finding a well-founded ordering or equivalently, finding a quantity which decreases at each step of computation. A widely accepted outcome of research undertaken in this direction dictates that the choice of ordering may have a surprising effect on efficiency (see [55, 56], and endeavours of the research group, University of St. Andrews, led by Martin *et al*, especially on classification of orderings). In the course of time, Multiset Path Orderings (i.e. Recursive Path Orders with mulitset status only) [25, 27], Simplification Orders [53] and [81], Lexicographic Path Orders [13], Decomposition Orders [41], etc. including Higher-order Path Orderings, have been experimented (see [18], [13], and [40] for various details).

In [40], it is shown that a termination proof for a TRS using multiset path orderings yields a primitive recursive bound on the length of derivations. The striking point is that this result holds for a great variety of path orderings, including $AC-$ Path Orderings described in [14], [44], [49] and [48] if lexicographic status is not incorporated. The reference [50] is a comprehensive work on Termination, $AC-$ Termination and Dependency Pairs (introduced in [1]).

The study around finding bounds on derivation length does have an antecedent in associating ordinals to proofs that computations terminate *e la Turing* (1994). This is now generally known as Floyd's method of analyzing program correctness. An ordinal technique (including ordinal powers) was exploited to prove termination of an example which calculates factorial by repeated addition. Ordinals enable us to link termination proofs with classical proof and recursion theory.

The specification of an abstract data type (ADT) for msets has been studied in ([6], pp. 1–29). Dovier, Policriti and Rossi ([29], pp. 208–234), after developing mathematics of msets, substantiate that msets are the fundamental data structures for $P$ systems. In fact, in computational sense, an mset is just a data structure, which differs from a set in permitting repetition of some of its elements and that from a *list* in being unordered, and hence it turns out to be a suitable modelling tool for a large class of real-life phenomena. Ross and Stoyanovich [71], study cardinality bounded msets in Database systems to overcome consistency and performance problems that conventional representations in relational database suffered from.

Petrovsky ([67], pp. 174–184) explores a new dimension. He, after providing an extensive treatment of mset operations, discusses "Cluster" analysis in mset spaces, and demonstrates their application in augmenting Decision Support System (DSS). It is shown that "a multi-aspect analysis of problem and structuring alternatives allow us to gain an insight into the problem nature and find better decisions. " ... suggested the tools for structuring a collection of objects represented by many qualitative attributes when a lot of copies of objects or values of attributes describing them exist. ... approach is based on a theory of multiset metric spaces and indexes of difference/similarity between multisets".

A "rough" set approach to multi–attribute decision analysis of Pawlak and

Slowinski ([70] , pp. 443–459) is a pioneering work in this direction.

## 4. Concluding remarks and some future directions

Multiset theory, owing to its multitudinal applications, has occupied a central place among a number of non-classical (nonstandard) set theories developed during the last few decades.

Category theory is emerging as a strong alternative: Epsilons are replaced by "Arrows", but that time is not yet ripe. Currently, all efforts seem to point to discovering some primitive theory of structures (a pre-set theoretic) to which all set theories can be alluded (see [10], pp. 321-322, for details).

### 4.1. Some future directions

1. Pursuing the techniques followed in [82], an extension of the program KRIPKE can be achieved by interpreting a multiset of formulae viz., $[A, B^m, C^n, \ldots]$ as an 'Ordinal' tree viz., $[A.B^m.C^n.\ldots]$, where $A$, $B$, $C$, … stand for 'events' and $A^k$ for '$A$ occurs $k$ times'. ([76] , p. 114)

2. A reasonable expectation is that a set theory based on $\omega$–valued logic ("$x \in y$" can have a truth value $n \in \omega$) and a multiset theory based on classical $2-$valued logic ('$x \in^n y$ can have only two truth values) are equivalent theories ([10], pp. 337–338). See Skolem ([77], Chapter 18): "… it seems to be possible to obtain a consistent set theory with an unrestricted axiom of comprehension if all rational numbers $\geq 0$ and $\leq 1$ are allowed as truth values" (p. 69).

3. "Multiset as a model for Multi-Attribute objects" is expected to play a significant role in the area of mathematical modelling, Discovery of Intelligent systems, control of Non-linear mechanical systems, just to mention a few (see [67]).

4. Discovering competing simulators for biological systems seems to dominate the research in biotechnology for a foreseeable future (see [12], [47], [24] and [4]).

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