A computational proof of Huang's degree theorem

(Don Knuth, Stanford Computer Science Department, 28 July 2019)

Hao Huang recently posted his proof [1] of a beautiful combinatorial theorem that establishes Nisan and Szegedy's 30-year-old Sensitivity Conjecture for Boolean functions:

Theorem. *Any set H of $2^{n-1}+1$ vertices of the n-cube contains a vertex with at least $\sqrt{n}$ neighbors in H.*

His proof used the interesting sequence of symmetric $2^n \times 2^n$ matrices A_n defined recursively by

$$A_0 = (0); \qquad A_n = \begin{pmatrix} A_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -A_{n-1} \end{pmatrix}, \quad \text{for } n > 0.$$

An easy induction proves that $A_n^2 = nI_{2^n}$ for all $n \geq 0$. Furthermore, every row and column of A_n has exactly n nonzero entries. Indeed, if we number the rows and columns in binary notation from $0\ldots0$ to $1\ldots1$, the entry in row $\alpha = a_1 \ldots a_n$ and column $\beta = b_1 \ldots b_n$ is ±1 when $|a_1 - b_1| + \cdots + |a_n - b_n| = 1$, otherwise it is zero.

Now let B_n be the $2^n \times 2^{n-1}$ matrix

$$B_n = \begin{pmatrix} A_{n-1} + \sqrt{n}\, I_{2^{n-1}} \\ I_{2^{n-1}} \end{pmatrix}.$$

Then B_n has rank 2^{n-1}, and we have

$$A_n B_n = \begin{pmatrix} A_{n-1}^2 + \sqrt{n} A_{n-1} + I_{2^{n-1}} \\ \sqrt{n}\, I_{2^{n-1}} \end{pmatrix} = \begin{pmatrix} \sqrt{n} A_{n-1} + nI_{2^{n-1}} \\ \sqrt{n}\, I_{2^{n-1}} \end{pmatrix} = \sqrt{n}\, B_n.$$

If B^* denotes the $2^{n-1} - 1$ rows of B_n that do not belong to H, we can find a unit $2^{n-1} \times 1$ vector x such that $B^* x = 0$. [That's $2^{n-1} - 1$ homogenous linear equations in 2^{n-1} variables.] Then $y = B_n x$ is a $2^n \times 1$ vector that's zero outside of H; and $A_n y = \sqrt{n} y$.

Let α be an index such that $|y_\alpha| = \max\{|y_0|, \ldots, |y_{2^n-1}|\}$. Then

$$|\sqrt{n} y_n| = |A_n y| = \Bigl|\sum_{\beta=0}^{2^n-1} A_{n\alpha\beta} y_\beta\Bigr| \leq \sum_{\beta \in H} |A_{n\alpha\beta}||y_\alpha| = |y_\alpha| \sum_{\beta \in H} [\alpha \text{ is adjacent to } \beta].$$

In other words, α has at least $\sqrt{n}$ neighbors β in H. QED.

Notes. This proof essentially fleshes out the idea that Shalev Ben-David contributed on July 3 to Scott Aaronson's blog [2]. Another basis for the "positive" eigenvectors is

$$C_n = \begin{pmatrix} I_{2^{n-1}} \\ \sqrt{n}\, I_{2^{n-1}} - A_{n-1} \end{pmatrix}.$$

I thought at first that a tricky combination of the columns of B_n and C_n might make the proof *really* simple; but that idea didn't pan out.

If $\alpha = a_1 \ldots a_n$ is adjacent to $\beta = b_1 \ldots b_n$ by complementing coordinate j, the sign of $A_{n\alpha\beta}$ is + if and only if $a_1 + \cdots + a_{j-1}$ is even.

Vitor Bosshard has pointed out in [2] that Huang's matrices are rather like the skew-symmetric adjacency matrices of the Klee–Minty cube:

$$\widehat{A}_0 = (0); \qquad \widehat{A}_n = \begin{pmatrix} \widehat{A}_{n-1} & I_{2^{n-1}} \\ -I_{2^{n-1}} & -\widehat{A}_{n-1} \end{pmatrix}, \quad \text{for } n > 0.$$

The corresponding eigenvectors for eigenvalue $\sqrt{n}i$ have similar bases $\widehat{B}_n$ and $\widehat{C}_n$. A Klee–Minty arc is directed from α to β if and only if $a_1 + \cdots + a_j$ is even.

[1] www.mathcs.emory.edu/~hhuan30/papers/sensitivity_1.pdf
[2] www.scottaaronson.com/blog/?p=4229