Harnack Inequalities.
An Introduction.

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Abstract

The aim of this article is to give an introduction to certain inequalities named after Carl Gustav Axel von Harnack. These inequalities were originally defined for harmonic functions in the plane and much later became an important tool in the general theory of harmonic functions and partial differential equations. We restrict ourselves mainly to the analytic perspective but comment on the geometric and probabilistic significance of Harnack inequalities. Our focus is classical results rather than latest developments. We give many references to this topic but emphasize that neither the mathematical story of Harnack inequalities nor the list of references given here is complete.

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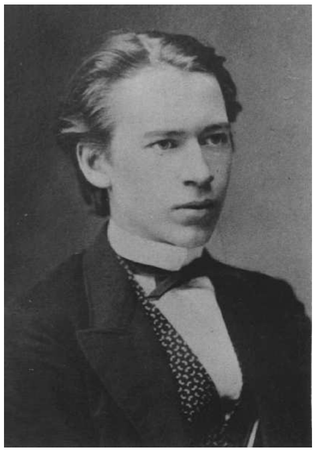
1 Carl Gustav Axel von Harnack

On May 7, 1851 the twins Carl Gustav Adolf von Harnack and Carl Gustav Axel von Harnack are born in Dorpat, which at that time is under German influence and is now known as the Estonian university city Tartu. Their father Theodosius von Harnack (1817-1889) works as a theologian at the university. The present article is concerned with certain inequalities derived by the mathematician Carl Gustav Axel von Harnack who died on April 3, 1888 as a professor of mathematics at the Polytechnikum in Dresden. His short life is devoted to science in general, mathematics and teaching in particular. For a mathematical obituary including a complete list of Harnack's publications we refer the reader to [Vos88].



C. G. Axel von Harnack (1851-1888) [1]

Carl Gustav Axel von Harnack is by no means the only family member working in science. His brother, Carl Gustav Adolf von Harnack becomes a famous theologian and professor of ecclesiastical history and pastoral theology. Moreover, in 1911 Adolf von Harnack becomes the founding president of the Kaiser-Wilhelm-Gesellschaft which today is called the Max Planck society. That is why the highest award of the Max Planck society is the Harnack medal.

After studying at the university of Dorpat[2] Axel von Harnack moves to Erlangen in 1873 where he becomes a student of Felix Klein. He knows Erlangen from the time his father was teaching there. Already in 1875, he publishes his PhD thesis[3] entitled "Ueber die Verwerthung der elliptischen Funktionen für die Geometrie der Curven dritter Ordnung." He is strongly influenced by the works of Alfred Clebsch and Paul Gordan[4] and is supported by the latter.

In 1875 Harnack receives the so-called "venia legendi"[5] from the university of Leipzig. One year later he accepts a position at the Technical University Darmstadt. In 1877 Harnack marries Elisabeth von Oettingen from a village close to Dorpat. They move to Dresden where Harnack takes a position at the Polytechnikum, which becomes a technical university in 1890.

In Dresden his main task is to teach calculus. In several talks Harnack develops his

[1]Photograph courtesy of Prof. em. Dr. med. Gustav Adolf von Harnack, Düsseldorf

[2]His thesis from 1872 on series of conic sections was not published.

[3]Math. Annalen, Vol. 9, 1875, 1-54

[4]such as: A. Clebsch, P. Gordan, *Theorie der Abelschen Funktionen*, 1866, Leipzig

[5]A credential permitting to teach at a university, awarded after attaining a habilitation.

own view of what the job of a university teacher should be: *clear and complete treatment of the basic terminology, confinement of the pure theory and of applications to evident problems, precise statements of theorems under rather strong assumptions.*[6] From 1877 on, Harnack shifts his research interests towards analysis. He works on function theory, Fourier series, and the theory of sets. At the age of 36 he has published 29 scientific articles and is well known among his colleagues in Europe. From 1882 on he suffers from health problems which force him to spend long periods in a sanatorium.

Harnack writes a textbook[7] which receives a lot of attention. During a stay of 18 months in a sanatorium in Davos he translates the "Cours de calcul différentiel et intégral" of J.-A. Serret[8] adding several long and significant comments. In his last years Harnack works on potential theory. His book [Har87] is the starting point of a rich and beautiful story: Harnack inequalities.

2 The classical Harnack inequality

On page 62 in paragraph 19 of [Har87] Harnack formulates and proves the following theorem in the case $d = 2$.

Theorem 2.1 *Let $u : B_R(x_0) \subset \mathbb{R}^d \to \mathbb{R}$ be a harmonic function which is either non-negative or non-positive. Then the value of u at any point in $B_r(x_0)$ is bounded from above and below by the quantities*

$$u(x_0)\Big(\frac{R}{R+r}\Big)^{d-2}\frac{R-r}{R+r} \quad \textit{and} \quad u(x_0)\Big(\frac{R}{R-r}\Big)^{d-2}\frac{R+r}{R-r}\,. \tag{2.1}$$

The constants above are scale invariant in the sense that they do not change for various choices of R when $r = cR$, $c \in (0,1)$, fixed. In addition, they do neither depend on the position of the ball $B_R(x_0)$ nor on u itself. The assertion holds for any harmonic function and any ball $B_R(x_0)$. We give the standard proof for arbitrary $d \in \mathbb{N}$ using the Poisson formula. The same proof allows to compare $u(y)$ with $u(y')$ for $y, y' \in B_r(x_0)$.

[6] Heger, Reidt (eds.), *Handbuch der Mathematik*, Breslau 1879 and 1881

[7] *Elemente der Differential- und Integralrechnung*, 400 pages, 1881, Leipzig, Teubner

[8] 1867-1880, Paris, Gauthier-Villars

Proof: Let us assume that u is non-negative. Set $\rho = |x - x_0|$ and choose $R' \in (r, R)$. Since u is continuous on $\overline{B_{R'}(x_0)}$ the Poisson formula can be applied, i.e.

$$u(x) = \frac{R'^2 - \rho^2}{\omega_d R'} \int\limits_{\partial B_{R'}(x_0)} u(y)|x-y|^{-d}\, dS(y)\,. \tag{2.2}$$

Note

$$\frac{R'^2 - \rho^2}{(R' + \rho)^d} \;\leq\; \frac{R'^2 - \rho^2}{|x-y|^d} \;\leq\; \frac{R'^2 - \rho^2}{(R' - \rho)^d}\,. \tag{2.3}$$

Combining (2.2) with (2.3) and using the mean value characterization of harmonic functions we obtain

$$u(x_0)\Big(\frac{R'}{R'+\rho}\Big)^{d-2}\frac{R'-\rho}{R'+\rho} \leq u(x) \leq u(x_0)\Big(\frac{R'}{R'-\rho}\Big)^{d-2}\frac{R'+\rho}{R'-\rho}\,.$$

Considering $R' \to R$ and realizing that the bounds are monotone in ρ inequality (2.1) follows. The theorem is proved. ■

Although the Harnack inequality (2.1) is almost trivially derived from the Poisson formula the consequences that may be deduced from it are both deep and powerful. We give only four of them here.

1. If $u : \mathbb{R}^d \to \mathbb{R}$ is harmonic and bounded from below or bounded from above then it is constant. (Liouville Theorem).
2. If $u : \{x \in \mathbb{R}^3; 0 < |x| < R\} \to \mathbb{R}$ is harmonic and satisfies $u(x) = o(|x|^{2-d})$ for $|x| \to 0$ then $u(0)$ can be defined in such a way that $u : B_R(0) \to \mathbb{R}$ is harmonic. (Removable Singularity Theorem).
3. Let $\Omega \subset \mathbb{R}^d$ be a domain and (g_n) be a sequence of boundary values $g_n : \partial\Omega \to \mathbb{R}$. Let (u_n) be the sequence of corresponding harmonic functions in Ω. If g_n converges uniformly to g then u_n converges uniformly to u. The function u is harmonic in Ω with boundary values g. (Harnack's first convergence theorem).
4. Let $\Omega \subset \mathbb{R}^d$ be a domain and (u_n) be a sequence of monotonically increasing harmonic functions $u_n : \Omega \to \mathbb{R}$. Assume that there is $x_0 \in \Omega$ with $|u_n(x_0)| \leq K$ for all n. Then u_n converges uniformly on each subdomain $\Omega' \Subset \Omega$ to a harmonic function u. (Harnack's second convergence theorem).

There are more consequences such as results on gradients of harmonic functions. The author of this article is not able to judge when and by whom the above results were proved first in full generality. Let us shortly review some early contributions to the theory of Harnack inequalities and Harnack convergence theorems. Only three years after [Har87] is published Poincaré makes substantial use of Harnack's results in the celebrated paper [Poi90]. The first paragraph of [Poi90] is devoted to the study of the Dirichlet problem in three dimensions and the major tools are Harnack inequalities.

Lichtenstein [Lic12] proves a Harnack inequality for elliptic operators with differentiable coefficients and including lower order terms in two dimensions. Although the methods applied are restricted to the two-dimensional case the presentation is very modern. In [Lic13] he proves the Harnack's first convergence theorem using Green's functions. As Feller [Fel30] remarks this approach carries over without changes to any space dimension $d \in \mathbb{N}$. Feller [Fel30] extends several results of Harnack and Lichtenstein. Serrin [Ser56] reduces the assumptions on the coefficients substantially. In two dimensions [Ser56] provides a Harnack inequality in the case where the leading coefficients are merely bounded; see also [BN55] for this result.

A very detailed survey article on potential theory up to 1917 is [Lic18][9]. Paragraphs 16 and 26 are devoted to Harnack's results. There are also several presentations of these results in textbooks; see as one example chapter 10 in [Kel29]. Kellogg formulates the Harnack inequality in the way it is used later in the theory of partial differential equations.

Corollary 2.2 *For any given domain $\Omega \subset \mathbb{R}^d$ and subdomain $\Omega' \Subset \Omega$ there is a constant $C = C(d, \Omega', \Omega) > 0$ such that for any non-negative harmonic function $u : \Omega \to \mathbb{R}$*

$$\sup_{x \in \Omega'} u(x) \leq C \inf_{x \in \Omega'} u(x) \,. \tag{2.4}$$

Before talking about Harnack inequalities related to the heat equation we remark that Harnack inequalities still hold when the Laplace operator is replaced by some fractional power of the Laplacian. More precisely, the following result holds.

Theorem 2.3 *Let $\alpha \in (0, 2)$ and $C(d, \alpha) = \frac{\alpha \Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha} \pi^{\frac{d}{2}} \Gamma(1-\frac{\alpha}{2})}$[10]. Let $u : \mathbb{R}^d \to \mathbb{R}$ be a non-negative function satisfying*

$$-(-\Delta)^{\alpha/2} u(x) = C(d, \alpha) \lim_{\varepsilon \to 0} \int\limits_{|h| > \varepsilon} \frac{u(x+h) - u(x)}{|h|^{d+\alpha}} \, dh = 0 \quad \forall \, x \in B_R(0) \,.$$

[9]Most articles refer wrongly to the second half of the third part of volume II, *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen.* The paper is published in the first half, though.

[10]$C(d, \alpha)$ is a normalizing constant which is important only when considering $\alpha \to 0$ or $\alpha \to 2$.

Then for any $y, y' \in B_R(0)$

$$u(y) \leq \Big|\frac{R^2 - |y|^2}{R^2 - |y'|^2}\Big|^{\alpha/2} \Big|\frac{R - |y|}{R + |y'|}\Big|^{-d} u(y') \,. \tag{2.5}$$

Poisson formulae for $(-\Delta)^{\alpha/2}$ are proved in [Rie38]. The above result and its proof can be found in chapter IV, paragraph 5 of [Lan72]. First, note that the above inequality reduces to (2.1) in the case $\alpha = 2$. Second, note a major difference: here the function u is assumed to be non-negative in all of $\mathbb{R}^d$. This is due to the non-local nature of $(-\Delta)^{\alpha/2}$. Harnack inequalities for fractional operators are currently studied a lot for various generalizations of $(-\Delta)^{\alpha/2}$. The interest in this field is due to the fact that these operators generate Markov jump processes in the same way $\frac{1}{2}\Delta$ generates the Brownian motion and $\sum\limits_{i,j=1}^{d} a_{ij}(.)D_iD_j$ a diffusion process. Nevertheless, in this article we restrict ourselves to a survey on Harnack inequalities for local differential operators.

It is not obvious what should be/could be the analog of (2.1) when considering non-negative solutions of the heat equation. It takes almost seventy years after [Har87] before this question is tackled and solved independently by Pini [Pin54] and Hadamard [Had54]. The sharp version of the result that we state here is taken from [Mos64], [AB94]:

Theorem 2.4 *Let $u \in C^\infty((0,\infty) \times \mathbb{R}^d)$ be a non-negative solution of the heat equation, i.e. $\frac{\partial}{\partial t}u - \Delta u = 0$. Then*

$$u(t_1, x) \leq u(t_2, y) \left(\tfrac{t_2}{t_1}\right)^{d/2} e^{\frac{|y-x|^2}{4(t_2-t_1)}}, \quad x, y \in \mathbb{R}^d, t_2 > t_1 \,., \tag{2.6}$$

The proof given in [AB94] uses results of [LY86] in a tricky way. There are several ways to reformulate this result. Taking the maximum and the minimum on cylinders one obtains

$$\sup_{|x|\leq\rho, \theta_1^- < t < \theta_2^-} u(t,x) \leq c \inf_{|x|\leq\rho, \theta_1^+ < t < \theta_2^+} u(t,x) \tag{2.7}$$

for non-negative solutions to the heat equation in $(0, \theta_2^+) \times B_R(0)$ as long as $\theta_2^- < \theta_1^+$. Here, the positive constant c depends on $d, \theta_1^-, \theta_2^-, \theta_1^+, \theta_2^+, \rho, R$. Estimate (2.7) can be illuminated as follows. Think of $u(t,x)$ as the amount of heat at time t in point x. Assume $u(t,x) \geq 1$ for some point $x \in B_\rho(0)$ at time $t \in (\theta_1^-, \theta_2^-)$. Then, after some waiting time, i.e. for $t > \theta_1^+$ $u(t,x)$ will be greater some constant c in all of the ball $B_\rho(0)$. It is necessary to wait some little amount of time for the phenomenon to occur since there is a sequence of solutions u_n satisfying $\frac{u_n(1,0)}{u_n(1,x)} \to 0$ for $n \to \infty$; see [Mos64]. As we see, already the statement of the parabolic Harnack inequality is much more subtle than its elliptic version.

3 Partial differential operators and Harnack inequalities

The main reason why research on Harnack inequalities is carried out up to today is that they are stable in a certain sense under perturbations of the Laplace operator. For example, inequality (2.4) holds true for solutions to a wide class of partial differential equations.

3.1 Operators in divergence form

In this section we review some important results in the theory of partial differential equations in divergence form. Suppose $\Omega \subset \mathbb{R}^d$ is a bounded domain. Assume that $x \mapsto A(x) = \big(a_{ij}(x)\big)_{i,j=1,\dots,d}$ satisfies $a_{ij} \in L^\infty(\Omega)$ $(i, j = 1, \dots, d)$ and

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^d \tag{3.1}$$

for some $\lambda > 0$. Here and below we use Einstein's summation convention. We say that $u \in H^1(\Omega)$ is a subsolution of the uniformly elliptic equation

$$-\operatorname{div}\big(A(.)\nabla u\big) = -D_i\big(a_{ij}(.)D_j u\big) = f \tag{3.2}$$

in Ω if

$$\int\limits_\Omega a_{ij} D_i u D_j \varphi \leq \int\limits_\Omega f\varphi \qquad \text{for any } \varphi \in H_0^1(\Omega), \varphi \geq 0 \text{ in } \Omega\,. \tag{3.3}$$

Here, $H^1(\Omega)$ denotes the Sobolev space of all $L^2(\Omega)$ functions with generalized first derivatives in $L^2(\Omega)$. The notion of supersolution is analogous. A function $u \in H^1(\Omega)$ satisfying $\int\limits_\Omega a_{ij} D_i u D_j \varphi = \int\limits_\Omega f\varphi$ for any $\varphi \in H_0^1(\Omega)$ is called a weak solution in Ω. Let us summarize Moser's results [Mos61] omitting terms of lower order.

Theorem 3.1 ([Mos61]) *Let $f \in L^q(\Omega)$, $q > d/2$.*

Local Boundedness: *For any non-negative subsolution $u \in H^1(\Omega)$ of (3.2) and any $B_R(x_0) \Subset \Omega$, $0 < r < R$, $p > 0$*

$$\sup_{B_r(x_0)} u \leq c\Big\{(R-r)^{-d/p}\|u\|_{L^p(B_R(x_0))} + R^{2-\frac{d}{q}}\|f\|_{L^q(B_R(x_0))}\Big\}\,, \tag{3.4}$$

where $c = c(d, \lambda, p, q)$ is a positive constant.

Weak Harnack Inequality: *For any non-negative supersolution $u \in H^1(\Omega)$ of (3.2) and any $B_R(x_0) \Subset \Omega$, $0 < \theta < \rho < 1$, $0 < p < \frac{n}{n-2}$*

$$\inf_{B_{\theta R(x_0)}} u + R^{2-\frac{d}{q}} \|f\|_{L^q(B_R(x_0))} \geq c\Big\{ R^{-d/p} \|u\|_{L^p(B_{\rho R}(x_0))} \Big\} . \tag{3.5}$$

where $c = c(d, \lambda, p, q, \theta, \rho)$ is a positive constant.

Harnack Inequality: *For any non-negative weak solution $u \in H^1(\Omega)$ of (3.2) and any $B_R(x_0) \Subset \Omega$*

$$\sup_{B_{R/2}(x_0)} u \leq c\Big\{ \inf_{B_{R/2}(x_0)} u + R^{2-\frac{d}{q}} \|f\|_{L^q(B_R(x_0))} \Big\} . \tag{3.6}$$

where $c = c(d, \lambda, q)$ is a positive constant.

Let us comment on the proofs of the above results. Estimate (3.4) is proved already in [DG57] but we explain the strategy of [Mos61]. By choosing appropriate test functions one can derive an estimate of the type

$$\|u\|_{L^{s_2}(B_{r_2}(x_0))} \leq c \|u\|_{L^{s_1}(B_{r_1}(x_0))} , \tag{3.7}$$

where $s_2 > s_1$, $r_2 < r_1$ and c behaves like $(r_1 - r_2)^{-1}$. Since $\big(|B_r(x_0)|^{-1} \int_{B_r(x_0)} u^s\big)^{1/s} \to \sup_{B_r(x_0)} u$ for $s \to \infty$ a careful choice of radii r_i and exponents s_i leads to the desired result via iteration of the estimate above. This is the famous "Moser iteration". The test functions needed to obtain (3.7) are of the form $\varphi(x) = \tau^2(x) u^s(x)$ where τ is a cut-off function. Additional minor technicalities such as the possible unboundedness of u and the right-hand side f have to be taken care of.

The proof of (3.5) can be split into two parts. For simplicity we assume $x_0 = 0, R = 1$. Set $\widetilde{u} = u + \|f\|_{L^q} + \varepsilon$ and $v = \widetilde{u}^{-1}$. One computes that v is a non-negative subsolution to (3.2). Applying (3.4) gives for any $\rho \in (\theta, 1)$ and any $p > 0$

$$\sup_{B_\theta} \widetilde{u}^{-p} \leq c \int_{B_\rho} \widetilde{u}^{-p} \qquad \text{or, equivalently}$$

$$\inf_{B_\theta} \widetilde{u} \geq c\Big(\int_{B_\rho} \widetilde{u}^{-p} \Big)^{-1/p} = c\Big(\int_{B_\rho} \widetilde{u}^{p} \int_{B_\rho} \widetilde{u}^{-p} \Big)^{-1/p} \Big(\int_{B_\rho} \widetilde{u}^{p} \Big)^{1/p} ,$$

where $c = c(d, q, p, \lambda, \theta, \rho)$ is a positive constant. The key step is to show the existence of $p_0 > 0$ such that

$$\Big(\Big(\int_{B_\rho} \widetilde{u}^{p} \Big) \Big(\int_{B_\rho} \widetilde{u}^{-p} \Big) \Big)^{-1/p} \geq c \qquad \Leftrightarrow \qquad \Big(\int_{B_\rho} \widetilde{u}^{p} \Big)^{1/p} \leq c \Big(\int_{B_\rho} \widetilde{u}^{-p} \Big)^{-1/p} .$$

This estimate follows once one establishes for $\rho < 1$

$$\int\limits_{B_\rho} e^{p_0|w|} \leq c(d,q,\lambda,\rho)\,, \tag{3.8}$$

for $w = \ln\widetilde{u} - (|B_\rho|)^{-1} \int\limits_{B_\rho|} \ln\widetilde{u}$. Establishing (3.8) is the major problem in Moser's approach and it becomes even more difficult in the parabolic setting. One way to prove (3.8) is to use $\varphi = \widetilde{u}^{-1}\tau^2$ as a test function and show with the help of Poincaré's inequality $w \in$ BMO, where BMO consists of all L^1-functions with "bounded mean oscillation", i.e. one needs to prove

$$r^{-d} \int\limits_{B_r(y)} \big|w - w_{y,r}\big| \leq K \qquad \forall\, B_r(y) \subset B_1(0)\ ,$$

where $w_{y,r} = \frac{1}{|B_r(y)|} \int\limits_{B_r(y)} w$. Then the so-called John-Nirenberg inequality from [JN61] gives $p_0 > 0$ and $c = c(d) > 0$ with $\int\limits_{B_r(y)} e^{\frac{p_0}{K}|w-w_{y,r}|} \leq c(d) r^d$ and thus (3.8). Note that [DG57] uses the same test function $\varphi = \widetilde{u}^{-1}\tau^2$ when proving Hölder regularity. [BG72] gives an alternative proof avoiding this embedding result. But there is as well a direct method of proving (3.8). Using Taylor's formula it is enough to estimate the L^1-norms of $\frac{|p_0 w|^k}{k!}$ for large k. This again can be accomplished by choosing appropriate test functions. This approach is explained together with many details of Moser's and DeGiorgi's results in [HL97].

On one hand, inequality (3.6) is closely related to pointwise estimates on Green functions; see [GW82] and [BF02]. On the other hand, a very important consequence of Theorem 3.1 is the following a-priori estimate which is independently established in [DG57] and implicitely in [Nas58].

Corollary 3.2 *Let $f \in L^q(\Omega)$, $q > d/2$. There exist two constants $\alpha = \alpha(d,q,\lambda) \in (0,1)$, $c = c(d,q,\lambda) > 0$ such that for any weak solution $u \in H^1(\Omega)$ of (3.2) $u \in C^\alpha(\Omega)$ and for any $B_R \Subset \Omega$ and any $x,y \in B_{R/2}$*

$$|u(x) - u(y)| \leq cR^{-\alpha}|x-y|^\alpha \Big\{ R^{-d/2}\|u\|_{L^2(B_R)} + R^{2-\frac{d}{q}}\|f\|_{L^q(B_R)} \Big\}\,, \tag{3.9}$$

where $c = c(d,\lambda,q)$ is a positive constant.

DeGiorgi [DG57] proves the above result by identifying a certain class to which all possible solutions to (3.2) belong, the so-called DeGiorgi class, and he investigates this class carefully. DiBenedetto/Trudinger [DT84] and DiBenedetto [DiB89] are able to prove that all functions in the DeGiorgi class directly satisfy the Harnack inequality.

The author of this article would like to emphasize that already [Har87] contains the main idea to the proof of Corollary 3.2. At the end of paragraph 19 Harnack formulates and proves the following observation in the two-dimensional setting:

> *Let u be a harmonic function on a ball with radius r. Denote by D the oscillation of u on the boundary of the ball. Then the oscillation of u on a inner ball with radius $\rho < r$ is not greater than $\frac{4}{\pi}\arcsin(\frac{\rho}{r})D$.*

Interestingly, Harnack seems to be the first to use the auxiliary function $v(x) = u(x) - \frac{M+m}{2}$ where M denotes the maximum of u and m the minimum over a ball. The use of such functions is the key step when proving Corollary 3.2.

So far, we have been speaking of harmonic function or solutions to linear elliptic partial differential equations. One feature of Harnack inequalities as well as of Moser's approach to them is that linearity does not play an important role. This is discovered by Serrin [Ser64] and Trudinger [Tru67]. They extend Moser's results to the situation of nonlinear elliptic equations of the following type:

$$\operatorname{div} \mathbf{A}(., u, \nabla u) + B(., u, \nabla u) = 0 \quad \text{weakly in } \Omega, \quad u \in W^{1,p}_{\text{loc}}(\Omega),\, p > 1\,.$$

Here, it is assumed that with $\kappa_0 > 0$ and non-negative κ_1, κ_2

$$\begin{aligned}&\kappa_0|\nabla u|^p - \kappa_1 \leq \mathbf{A}(., u, \nabla u)\cdot\nabla u\,,\\ &|\mathbf{A}(., u, \nabla u)| + |B(., u, \nabla u)| \leq \kappa_2(1 + |\nabla u|^{p-1})\,.\end{aligned}$$

Actually, [Tru67] allows for a more general upper bound including important cases such as $-\Delta u = c|\nabla u|^2$. Note that the above equation generalizes the Poisson equation in several aspects. $\mathbf{A}(x, u, \nabla)$ may be nonlinear in ∇u and may have a nonlinear growth in $|\nabla u|$, i.e. the corresponding operator may be degenerate. In [Ser64], [Tru67] a Harnack inequality is established and Hölder regularity of solutions is deduced. Trudinger [Tru81] relaxes the assumptions so that the minimal surface equation which is not uniformly elliptic can be handled. A parallel approach to regularity questions of nonlinear elliptic problems using the ideas of DeGiorgi but avoiding Harnack's inequality is carried out by Ladyzhenskaya/Uralzeva; see [LU68] and the references therein.

It is mentioned above that Harnack inequalities for solutions of the heat equation are more complicated in their formulation as well as in the proofs. This does not change when considering parabolic differential operators in divergence form. Besides the important articles [Pin54], [Had54] the most influential contribution is made by Moser in [Mos64], [Mos67], [Mos71]. Assume $(t, x) \mapsto A(t, x) = \big(a_{ij}(t, x)\big)_{i,j=1,\ldots,d}$ satisfies $a_{ij} \in L^\infty((0, \infty) \times \mathbb{R}^d)$ $(i, j = 1, \ldots, d)$ and for some $\lambda > 0$

$$\lambda|\xi|^2 \leq a_{ij}(t, x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^d, \xi \in \mathbb{R}^d\,. \tag{3.10}$$

Theorem 3.3 ([Mos64, Mos67, Mos71]) *Assume* $u \in L^\infty\big(0,T;L^2(B_R(0))\big) \cap L^2\big(0,T;H^1(B_R(0))\big)$ *is a non-negative weak solution to the equation*

$$u_t - \operatorname{div}\big(A(.,.)\nabla u\big) = 0 \quad \textit{in } (0,T)\times B_R(0)\,. \tag{3.11}$$

Then for any choice of constants $0 < \theta_1^- < \theta_2^- < \theta_1^+ < \theta_2^+$, $0 < \rho < R$ *there exists a positive constant* c *depending only on these constants and on the space dimension* d *such that (2.7) holds.*

Note that both "sup" and "inf" in (2.7) are to be understood as essential supremum and essential infimum respectively. As in the elliptic case a very important consequence of the above result is that bounded weak solutions are Hölder-continuous in the interior of the cylindrical domain $(0,T)\times B_R(0)$; see Theorem 2 of [Mos64] for a precise statement. The original proof given in [Mos64] contains a faulty argument in Lemma 4, this is corrected in [Mos67]. The major difficulty in the proof is, similar to the elliptic situation, the application of the so-called John-Nirenberg embedding. In the parabolic setting this is particularly complicated. In [Mos71] the author provides a significantly simpler proof by bypassing this embedding using ideas from [BG72]. Fabes and Garofalo [FG85] study the parabolic BMO space and provide a simpler proof to the embedding needed in [Mos64].

Ferretti and Safonov [FS01], [Saf02] propose another approach to Harnack inequalities in the parabolic setting. Their idea is to derive parabolic versions of mean value theorems implying growth lemmas for operators in divergence form as well as in non-divergence form[11].

Aronson [Aro67] applies Theorem 3.3 and proves sharp bounds on the fundamental solution $\Gamma(t,x;s,y)$ to the operator $\partial_t - \operatorname{div}\big(A(.,.)\nabla\big)$:

$$c_1(t-s)^{-d/2}e^{\frac{-c_2|x-y|^2}{|t-s|}} \le \Gamma(t,x;s,y) \le c_3(t-s)^{-d/2}e^{\frac{-c_4|x-y|^2}{|t-s|}}\,. \tag{3.12}$$

The constants $c_i > 0, i = 1,\ldots,4$, depend only on d and λ. It is mentioned above that Theorem 3.3 also implies Hölder-a-priori-estimates for solutions u of (3.11). At the time of [Mos64] these estimates are already well known due to the fundamental work of Nash [Nas58]. Fabes and Stroock [FS86] apply the technique of [Nas58] in order to prove (3.12). In other words, they use an assertion following from Theorem 3.3 in order to show another. This alone is already a major contribution. Moreover, they finally show that the results of [Nas58] already imply Theorem 3.3. See [FS84] for fine integrability results for the Green function and the fundamental solution.

Knowing extensions of Harnack inequalities from linear problems to nonlinear problems like [Ser64], [Tru67] it is a natural question whether such an extension is possible

[11] See Lemma 3.5 for the simplest version.

in the parabolic setting, i.e. for equations of the following type

$$u_t - \operatorname{div} \mathbf{A}(t, ., u, \nabla u) = B(t, ., u, \nabla) \qquad \text{in } (0, T) \times \Omega \,. \tag{3.13}$$

But the situation turns out to be very different for parabolic equations. Scale invariant Harnack inequalities can only be proved assuming linear growth of $\mathbf{A}$ in the last argument. First results in this direction are obtained parallely by Aronson/Serrin [AS67], Ivanov [Iva67], and Trudinger [Tru68]; see also [DiB93], [Iva82], [PÈ84]. For early accounts on Hölder regularity of solutions to (3.13) see [LSU68], [Kru61], [Kru68], [Kru69]. In a certain sense these results imply that the differential operator is not allowed to be degenerate or one has to adjust the scaling behavior of the Harnack inequality to the differential operator. The questions around this subtle topic are currently of high interest; we refer to results by Chiarenza/Serapioni [CS84], DiBenedetto [DiB88], the survey [DUV04] and latest achievements by DiBenedetto, Gianazza, Vespri [DGV06], [GV06a], [GV06b] for more information.

3.2 Degenerate operators

The title of this section is slightly confusing since degenerate differential operators like $\operatorname{div} \mathbf{A}(t, ., u, \nabla u)$ are already mentioned above. The aim of this section is to review Harnack inequalities for linear differential operators that do not satisfy (3.1) or (3.10). Again, the choice of results and articles mentioned is very selective. We present the general phenomenon and list related works at the end of the section.

Assume that $x \mapsto A(x) = \big(a_{ij}(x)\big)_{i,j=1,\dots,d}$ satisfies $a_{ji} = a_{ij} \in L^\infty(\Omega)$ $(i, j = 1, \dots, d)$ and

$$\lambda(x)|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda(x)|\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^d \,. \tag{3.14}$$

for some non-negative functions λ, Λ. As above we consider the operator $\operatorname{div}\big(A(.)\nabla u\big)$. Early accounts on the solvability of the corresponding degenerate elliptic equation together with qualitative properties of the solutions include [Kru63], [MS68], [MS71]. A Harnack inequality is proved in [EP72]. It is obvious that the behavior of the ratio $\Lambda(x)/\lambda(x)$ decides whether local regularity can be established or not. Fabes/Kenig/Serapioni [FKS82] prove a scale invariant Harnack inequality under the assumption $\Lambda(x)/\lambda(x) \le C$ and that λ belongs to the so-called Muckenhoupt class A_2, i.e. for all balls $B \subset \mathbb{R}^d$ the following estimate holds for a fixed constant $C > 0$:

$$\Big(\frac{1}{|B|}\int_B \lambda(x)\,dx\Big)\Big(\frac{1}{|B|}\int_B (\lambda(x))^{-1}\,dx\Big) \le C \,. \tag{3.15}$$

The idea is to establish inequalities of Poincaré type for spaces with weights where the weights belong to Muckenhoupt classes A_p and then to apply Moser's iteration technique. If $\Lambda(x)/\lambda(x)$ may be unbounded one cannot say in general whether a Harnack inequality or local Hölder *a priori* estimates hold. They may hold [Tru71] or may not [FSSC98]. Chiarenza/Serapioni [CS84],[CS87] prove related results in the parabolic set-up. Their findings include interesting counterexamples showing once more that degenerate parabolic operators behave much different from degenerate elliptic operators. Kruzhkov/Kolodii [KK77] do prove some sort of classical Harnack inequality for degenerate parabolic operators but the constant depends on other important quantities which makes it impossible to deduce local regularity of bounded weak solutions.

Assume that both λ, Λ satisfy (3.15) and the following doubling condition

$$\lambda(2B) \leq c\lambda(B), \qquad \Lambda(2B) \leq c\Lambda(B)^{12},$$

where $\lambda(M) = \int_M \lambda$ and $\Lambda(M) = \int_M \Lambda$. Then certain Poincaré and Sobolev inequalities hold with weights λ, Λ. Chanillo/Wheeden [CW86] prove a Harnack inequality of type (2.4) where the constant C depends on $\lambda(\Omega'), \Lambda(\Omega')$. For $\Omega = B_R(x_0)$ and $\Omega' = B_{R/2}(x_0)$ they discuss in [CW86] optimality of the arising constant C. In [CW88] a Green function corresponding to the degenerate operator is constructed and estimated pointwise under similar assumptions.

Let us list some other articles that deal with questions similar to the ones mentioned above.

Denerate elliptic operators: [CRS89] establishes a Harnack inequality, [Sal91] investigates Green's functions; [FGW94], [DCV96] allow for different new kinds of weights; [GL03] studies X-elliptic operators; [Moh02] further relaxes assumptions on the weights and allows for terms of lower order; [TW02] investigates quite general subelliptic operators in divergence form; [Zam02] studies lower order terms in Kato-Stummel classes; [FGM03] provides a new technique by by-passing the constructing of cut-off functions; [Fer06] proves a Harnack inequality for the two-weight subelliptic p-Laplacian.

Denerate parabolic operators: [GW90] establishes a Harnack inequality; [GW91] allows for time-dependent weights; [GW92] establishes bounds for the fundamental solution; [Ish99] allows for terms of lower order; [PP04] studies a class of hypoelliptic evolution equations.

[12] Here $\lambda(M)$, $\Lambda(M)$ stand for $\int\limits_M \lambda(x)\,dx$, $\int\limits_M \Lambda(x)\,dx$ respectively.

3.3 Operators in non-divergence form

A major breakthrough on Harnack inequalities (maybe the second one after Moser's works) is obtained by Krylov and Safonov in [KS79], [KS80], [Saf80]. They obtain parabolic and elliptic Harnack inequalities for partial differential operators in non-divergence form. We review their results without aiming at full generality. Assume $(t,x) \mapsto A(t,x) = \big(a_{ij}(t,x)\big)_{i,j=1,\ldots,d}$ satisfies (3.10). Set $Q_{\theta,R}(t_0,x_0) = (t_0+\theta R^2) \times B_R(x_0)$ and $Q_{\theta,R} = Q_{\theta,R}(0,0)$.

Theorem 3.4 ([KS80]) *Let $\theta > 1$ and $R \leq 2$, $u \in W_2^{1,2}(Q_{\theta,R})$, $u \geq 0$ be such that*

$$u_t - a_{ij}D_iD_ju = 0 \qquad a.e.\ in\ Q_{\theta,R}\,. \tag{3.16}$$

Then there is a constant C depending only on λ, θ, d such that

$$u(R^2,0) \leq Cu(\theta R^2, x) \quad \forall\, x \in B_{R/2}\,. \tag{3.17}$$

The constant C stays bounded as long as $(1-\theta)^{-1}$ and λ^{-1} stay bounded.

An important consequence of the above theorem are *a priori* estimates in the parabolic Hölder spaces for solutions u; see Theorem 4.1 in [KS80]. Hölder regularity results and a Harnack inequality for solutions to the elliptic equation $a_{ij}D_iD_ju = 0$ under the general assumptions above are proved first by Safonov in [Saf80]

In a certain sense these results extend research developed for elliptic equations in [Nir53, Cor56, Lan68]. Nirenberg proves Hölder regularity for solutions u in two dimensions. In higher dimensions he imposes a smallness condition on the quantity $\sum\limits_{i,j} \big(a_{ij}(x) - \delta_{ij}(x)\big)^2$; see [Nir54]. Cordes[Cor56] relaxes the assumptions and Landis [Lan68], [Lan71] proves Harnack inequalities but still requires the dispersion of eigenvalues of A to satisfy a certain smallness. Note that [Nir53] and [Cor56] additionally explain how to obtain $C^{1,\alpha}$-regularity of u from Hölder regularity which is important for existence results. The probabilistic technique developed by Krylov and Safonov in order to prove Theorem 3.4 resembles analytic ideas used in [Lan71][13]. The key idea is to prove a version of what Landis calls "growth lemma"[14]. Here is such a result in the simplest case:

Lemma 3.5 *Assume $\Omega \subset \mathbb{R}^d$ open and $z \in \Omega$. Suppose $|\Omega \cap B_R(z)| \leq \varepsilon |B_R(z)|$ for some $R > 0$, $\varepsilon \in (0,1)$. Then for any function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ with*

$$-\Delta u(x) \leq 0,\ 0 < u(x) \leq 1 \quad \forall x \in \Omega \cap B_R(z) \qquad \text{and } u(x) = 0\ \forall\, x \in \partial\Omega \cap B_R(z)$$

the estimate $u(z) \leq \varepsilon$ holds.

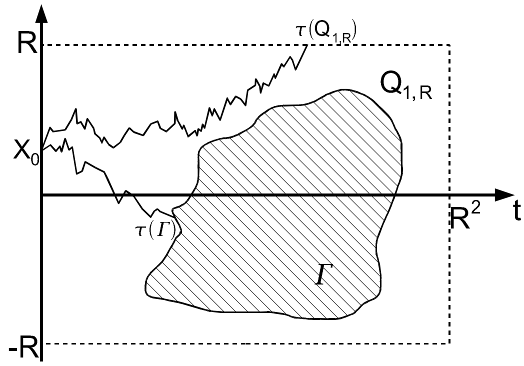
[13] Unfortunately the book was not translated until much later; see [Lan98].

[14] DiBenedetto [DiB89] refers to the same phenomenon as "expansion of positivity".

As just mentioned the original proof of Theorem 3.4 is probabilistic. Let us briefly explain the key ingredient of this proof. The technique involves hitting times of diffusion processes and implies a (analytic) result like Lemma 3.5 for quite general uniformly elliptic operators. One considers the diffusion process (X_t) associated to the operator $a_{ij}D_iD_j$ via the martingale problem. This process solves the following system of ordinary stochastic differential equations $dX_t = \sigma_t dB_t$. Here (B_t) is a $d-$dimensional Brownian motion and $\sigma_t^T\sigma_t = A$.

Assume that $\mathbb{P}(X_0 \leq \alpha R) = 1$ where $\alpha \in (0,1)$. Let $\Gamma \subset Q_{1,R}$ be a closed set satisfying $|\Gamma| \geq \varepsilon|Q_{1,R}|$ for some $\varepsilon > 0$. For a set $M \subset (0,\infty) \times \mathbb{R}^d$ let us denote the time when (X_t) hits the boundary ∂M by $\tau(M) = \inf\{t > 0; (t, X_t) \in \partial M\}$. The key idea in the proof of [KS79] is to show that there is $\delta > 0$ depending only on $d, \lambda, \alpha, \varepsilon$ such that

$$\mathbb{P}\big(\tau(\Gamma) < \tau(Q_{1,R})\big) \geq \delta \quad \forall\, R \in (0,1)\,.$$



Hitting times for a diffusion. [15]

The Harnack inequality, Theorem 3.4 and its elliptic counterpart open up the modern theory of fully nonlinear elliptic and parabolic equations of the form

$$F(., D^2u) = 0 \quad \text{ in } \Omega\,,$$

where D^2u denotes the Hessian of u. Evans [Eva82] and Krylov [Kry82] prove interior $C^{2,\alpha}(\Omega)$-regularity, Krylov in [Kry83] also $C^{2,\alpha}(\overline{\Omega})$-regularity. The approaches are based on the use of Theorem 3.4; see also the presentation in [GT01]. Harnack inequalities are proved by Caffarelli [Caf89] for viscosity solutions of fully nonlinear equations; see also [CC95a].

4 Geometric and probabilistic significance

In this section let us briefly comment on the non-Euclidean situation. Whenever we write “Harnack inequality” or “elliptic Harnack inequality” without referring to a certain type of differential equation we always mean the corresponding inequality for non-negative harmonic functions, i.e. functions u satisfying $\Delta u = 0$ including cases where Δ is the Laplace-Beltrami operator on a manifold. Analogously, the expression “parabolic Harnack inequality” refers to non-negative solutions of the heat equation.

[15] Figure courtesy of R. Husseini, SFB 611, Bonn

Bombieri/Giusti [BG72] prove a Harnack inequality for elliptic differential equations on minimal surfaces using a geometric analysis perspective. [BG72] is also well-known for a technique that can replace the use of the John-Nirenberg lemma in Moser's iteration scheme; see the discussion above. The elliptic Harnack inequality is proved for Riemannian manifolds by Yau in [Yau75]. A major breakthrough, the parabolic Harnack inequality and differential versions of it for Riemannian manifolds with Ricci curvature bounded from below are obtained by Li/Yau in [LY86] with the help of gradient estimates. In addition, they provide sharp bounds on the heat kernel.

Fundamental work has been carried out proving Harnack inequalities for various geometric evolution equations such as the mean curvature flow of hypersurfaces and the Ricci flow of Riemannian metrics. We are not able to give details of these results here and refer the reader to the following articles: [Ham88, Cho91, Ham93, And94, Yau94, CC95b, Ham95, HY97, CH97, CCCY03]. Finally, we refer to [Mul06] for a detailed discussion of how the so called differential Harnack inequalitiy of [LY86] enters the work of G. Perelman.

The parabolic Harnack inequality is not only a property satisfied by non-negative solutions to the heat equation. It says a lot about the structure of the underlying manifold or space. Independently, Saloff-Coste [SC92] and Grigor'yan [Gri91] prove the following result. Let (M, g) be a smooth, geodesically complete Riemannian manifold of dimension d. Let $r_0 > 0$. Then the two properties

$$\big|B(x,2r)\big| \leq c_1 \big|B(x,r)\big|, \quad 0 < r < r_0, x \in M\,, \tag{4.1}$$

$$\int\limits_{B(x,r)} |f - f_{x,r}|^2 \leq c_2 r^2 \int\limits_{B(x,2r)} |\nabla f|^2, \quad 0 < r < r_0, x \in M, f \in C^\infty(M)\,, \tag{4.2}$$

together are equivalent to the parabolic Harnack inequality. (4.1) is called volume doubling condition and (4.2) is a weak version of Poincaré's inequality. Since both conditions hold for manifolds with Ricci curvature bounded from below these results imply several but not all results obtained in [LY86]. See also [SC95] for another presentation of this relation.

The program suggested by [SC92], [Gri91] is carried out by Delmotte in [Del99] in the case of locally finite graphs and by Sturm [Stu96] for time dependent Dirichlet forms on locally compact metric spaces including certain subelliptic operators. For both results a probabilistic point of view is more than helpful. An elliptic Harnack inequality is used by Barlow and Bass to construct a Brownian motion on the Sierpiński carpet in [BB93]; see [BB99] and [BB00] for related results. A parabolic Harnack inequality with a non-diffusive space-time scaling is proved [BB04] on infinite connected weighted graphs. Moreover it is shown that this inequality is stable under bounded transformations of the conductances.

It is interesting to note that the elliptic Harnack inequality is weaker than its parabolic counterpart. For instance, it does not imply (4.1) for all $x \in M$. It is not known which set of conditions is equivalent to the elliptic Harnack inequality. Even on graphs the situation can be difficile. The graph version of the Sierpińkski gasket, as one example, satisfies the elliptic Harnack inequality but not (4.2). Graphs with a bottleneck-structure again might satisfy the elliptic Harnack inequality but violate (4.1); see [Del02] for a detailed discussion of these examples and [HSC01], [Bar05] for recent progress in this direction.

5 Closing remarks

As pointed out in the abstract, this article is incomplete in many respects. It is concerned with Harnack inequalities for solutions of partial differential equations. Emphasis is placed on elliptic and parabolic differential equations that are non-degenerate. Degenerate operators are mentioned only briefly. Fully nonlinear operators, Schrödinger operators, and complex valued functions are not mentioned at all with only few exceptions. The same applies to boundary Harnack inequalities, systems of differential equations, and the interesting connection between Harnack inequalities and problems with free boundaries. In section 2 Harnack inequalities for nonlocal operators are mentioned only briefly although they attract much attention at present; see [BS05] and [BK05]. In the above presentation the parabolic Harnack inequality on manifolds is not treated according to its significance. Harmonic functions in discrete settings, i.e. on graphs or related to Markov chains are not dealt with; see [Ale02], [DSC95], [Dod84], [GT02], [Law91], [LP93] for various aspects of this field.

It would be a major and very interesting research project to give a complete account of all topics where Harnack inequalities are involved.

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