Calculus for mathematicians

D. J. Bernstein, University of Illinois at Chicago, `djb@pobox.com`

19970403, draft 5

1. Introduction

This booklet presents the main concepts, theorems, and techniques of single-variable calculus. It differs from a typical undergraduate real analysis text in that (1) it focuses purely on calculus, not on developing topology and analysis for their own sake; (2) it's short.

Notation and terminology. The reader must be comfortable with *functions*, not just numbers, as objects of study. I use the notation $x \mapsto x^2$ for the function that takes x to x^2; thus $(x \mapsto x^2)(3) = 9$. In general $f = (t \mapsto f(t))$ for any function f.

An **open ball around** c means an interval $\text{Ball}(c, h) = \{x : |x - c| < h\}$ for some positive real number h. The intersection of two open balls around c is another open ball around c.

If S is a set, and $f(x)$ is defined for all $x \in S$, then $f(S)$ is defined as $\{f(x) : x \in S\}$.

Part I. Continuity

2. Continuous functions

Definition 2.1. *Let f be a function defined at c. Then f is* **continuous at** *c if, for any open ball F around $f(c)$, there is an open ball B around c such that $f(B) \subseteq F$.*

In other words, if f is continuous at c, and F is an open ball around $f(c)$, then there is some $h > 0$ such that $f(x) \in F$ for all x with $|x - c| < h$.

Example: The function $x \mapsto 3x$ is continuous—i.e., continuous at c for every c. Indeed, $\text{Ball}(3c, \epsilon)$ contains $(x \mapsto 3x)(\text{Ball}(c, \epsilon/3))$, because $|x - c| < \epsilon/3$ implies $|3x - 3c| < \epsilon$.

Another example: If $f(x) = 3$ for $x < 2$ and $f(x) = 5$ for $x \ge 2$, then f is not continuous at 2. Indeed, consider the open ball $F = \text{Ball}(5, 1)$. If B is any open ball around 2, then B contains numbers smaller than 2, so $3 \in f(B)$; thus $f(B)$ is not contained in F.

Theorem 2.2. *Let f and g be functions continuous at c. Assume that $f(x) = g(x)$ for all $x \ne c$ such that $f(x)$ and $g(x)$ are both defined. Then $f(c) = g(c)$.*

Proof. I will show that $|f(c) - g(c)| < 2\epsilon$ for any $\epsilon > 0$. Write $F = \text{Ball}(f(c), \epsilon)$ and $G = \text{Ball}(g(c), \epsilon)$. By continuity of f and g, there are balls A and B around c such that $f(A) \subseteq F$ and $g(B) \subseteq G$. Find a point $x \ne c$ contained in both A and B. By construction $f(x) \in F$ and $f(x) = g(x) \in G$, so $|f(c) - g(c)| \le |f(x) - f(c)| + |f(x) - g(c)| < 2\epsilon$ as claimed. □

3. Continuity of sums, products, and compositions

Theorem 3.1. *Let f and g be functions continuous at c. Define $h = f + g$. Then h is continuous at c.*

Proof. Given a ball $H = \mathrm{Ball}(h(c), \epsilon)$, consider the balls $F = \mathrm{Ball}(f(c), \epsilon/2)$ and $G = \mathrm{Ball}(g(c), \epsilon/2)$. By continuity of f and g, there are open balls A and B around c such that $f(A) \subseteq F$ and $g(B) \subseteq G$. Define $D = A \cap B$; D is an open ball around c. If $x \in D$ then $f(x) \in F$ and $g(x) \in G$ so $h(x) = f(x) + g(x) \in H$. Thus $h(D) \subseteq H$. □

Theorem 3.2. *Let f and g be functions continuous at c. Define $h = fg$. Then h is continuous at c.*

Proof. Define $L = f(c)$ and $M = g(c)$, so that $LM = h(c)$. Given an open ball $H = \mathrm{Ball}(LM, \epsilon)$, I will find an open ball D around c so that $h(D) \subseteq H$.

If $L = M = 0$, take the intersection of open balls where $|f(x)| < \epsilon$ and $|g(x)| < 1$. Then $|h(x)| < \epsilon$.

If $L = 0$ and $M \neq 0$, take the intersection of open balls where $|f(x)| < \epsilon/(2\,|M|)$ and $|g(x) - M| < |M|$. Then $|g(x)| < 2\,|M|$ so $|h(x)| < \epsilon$. Similarly if $L \neq 0$ and $M = 0$.

If $L \neq 0$ and $M \neq 0$, take the intersection of open balls where $|f(x) - L| < \epsilon/(4\,|M|)$, $|g(x) - M| < \epsilon/(2\,|L|)$, and $|g(x) - M| < |M|$. Then $|g(x)| < 2\,|M|$ so $|h(x) - LM| = |g(x)(f(x) - L) + L(g(x) - M)| < 2\,|M|\,(\epsilon/(4\,|M|)) + |L|\,(\epsilon/(2\,|L|)) = \epsilon$. □

Theorem 3.3. *Let g be a function continuous at c. Let f be a function continuous at $g(c)$. Define $h = (x \mapsto f(g(x)))$. Then h is continuous at c.*

For example, $x \mapsto \cos 2x$ is continuous, since $x \mapsto 2x$ and $y \mapsto \cos y$ are continuous,

Proof. Let F be an open ball around $h(c) = f(g(c))$. By continuity of f, there is some open ball G around $g(c)$ with $f(G) \subseteq F$. By continuity of g, there is some open ball B around c with $g(B) \subseteq G$. Finally $h(B) = f(g(B)) \subseteq f(G) \subseteq F$. □

4. Continuity of simple functions

Theorem 4.1. *$x \mapsto b$ is continuous at c, for any b and c.*

Proof. $\mathrm{Ball}(b, h)$ contains $(x \mapsto b)(D)$ for any open ball D. □

Theorem 4.2. *$x \mapsto x$ is continuous at c, for any c.*

Proof. $\mathrm{Ball}(c, h)$ contains $(x \mapsto x)(\mathrm{Ball}(c, h))$. □

By Theorems 3.2 and 4.2, $x \mapsto x^2$ is continuous; $x \mapsto x^3$ is continuous; in general $x \mapsto x^n$ is continuous for any positive integer n. Thus, by Theorems 3.1, 3.2, and 4.1, any polynomial function $x \mapsto c_0 + c_1 x + \cdots + c_n x^n$ is continuous.

The function $x \mapsto 1/x$ is continuous at c for $c \neq 0$. (It's not even defined at 0, so it can't be continuous there.) By Theorem 3.3, $x \mapsto 1/f(x)$ is continuous whenever f is continuous and nonzero. For example, $x \mapsto x^n$ is continuous except at 0 when n is a negative integer.

Part II. Derivatives

5. Differentiable functions

Definition 5.1. *Let f be a function defined at c. Then f is* **differentiable at** *c if there is a function f_1, continuous at c, such that $f = (x \mapsto f(c) + (x - c)f_1(x))$.*

Definition 5.2. *Let f be a function defined at c. Then f* **has derivative** *d* **at** *c if there is a function f_1, continuous at c, such that $f = (x \mapsto f(c) + (x - c)f_1(x))$ and $f_1(c) = d$.*

By Theorem 2.2, there is at most one continuous function f_1 satisfying $f_1(x) = (f(x) - f(c))/(x - c)$ for all $x \neq c$, so f has at most one derivative at c, called **the derivative of f at** c. The derivative of f at c is written $f'(c)$. The **derivative of** f, written f', is the function $c \mapsto f'(c)$.

For example, consider the function $f = (x \mapsto x^2)$. Here $f(x) = f(3) + (x - 3)f_1(x)$ with $f_1 = (x \mapsto x + 3)$. The function f_1 is continuous at 3, so f is differentiable at 3; its derivative at 3 is $f_1(3) = 6$. In general $f'(c) = 2c$.

Theorem 5.3. *Let f be a function. If f is differentiable at c then f is continuous at c.*

Proof. By definition of differentiability, there is a function f_1, continuous at c, with $f = (x \mapsto f(c) + (x - c)f_1(x))$. Apply Theorems 3.1, 3.2, 4.1, and 4.2. □

6. Derivatives of sums, products, and compositions

Theorem 6.1. *Let f and g be functions. Define $h = f + g$. If f and g are differentiable at c then h is differentiable at c. Furthermore $h'(c) = f'(c) + g'(c)$.*

In short $(f + g)' = f' + g'$ if the right side is defined. This is the **sum rule**.

Proof. Say $f(x) = f(c) + (x - c)f_1(x)$ and $g(x) = g(c) + (x - c)g_1(x)$ with f_1 and g_1 continuous at c. Define $h_1 = f_1 + g_1$; then h_1 is continuous at c by Theorem 3.1, and $h(x) = h(c) + (x - c)h_1(x)$, so h is differentiable at c. Finally $h'(c) = h_1(c) = f_1(c) + g_1(c) = f'(c) + g'(c)$. □

Theorem 6.2. *Let f and g be functions. Define $h = fg$. If f and g are differentiable at c then h is differentiable at c. Furthermore $h'(c) = f'(c)g(c) + f(c)g'(c)$.*

In short $(fg)' = f'g + fg'$ if the right side is defined. This is the **product rule**.

Proof. Say $f(x) = f(c) + (x - c)f_1(x)$ and $g(x) = g(c) + (x - c)g_1(x)$ with f_1 and g_1 continuous at c. Then $h(x) = h(c) + (x - c)h_1(x)$ where $h_1(x) = f_1(x)g(x) + f(c)g_1(x)$. This function h_1 is continuous at c by Theorems 3.1, 3.2, 4.1, and 5.3, so h is differentiable at c, with derivative $h_1(c) = f_1(c)g(c) + f(c)g_1(c) = f'(c)g(c) + f(c)g'(c)$. □

Theorem 6.3. *Let f and g be functions. Define $h = (x \mapsto f(g(x)))$. If g is differentiable at c, and f is differentiable at $g(c)$, then h is differentiable at c. Furthermore $h'(c) = f'(g(c))g'(c)$.*

In short $(f \circ g)' = (f' \circ g)g'$ if the right side is defined. This is the **chain rule**.

Proof. Write $b = g(c)$. Say $f(x) = f(b)+(x-b)f_1(x)$ and $g(x) = b+(x-c)g_1(x)$ with f_1 continuous at b and g_1 continuous at c. Now $h(x) = f(g(x)) = f(b)+(g(x)-b)f_1(g(x)) = f(b)+(x-c)g_1(x)f_1(g(x))$. Thus $h(x) = h(c)+(x-c)h_1(x)$ where $h_1(x) = g_1(x)f_1(g(x))$. Finally h_1 is continuous at c by Theorems 3.3, 3.2, and 5.3, so h is differentiable at c, with derivative $h_1(c) = g_1(c)f_1(g(c)) = g'(c)f'(g(c))$. □

7. Derivatives of simple functions

A constant function, such as $x \mapsto 17$, has derivative $c \mapsto 0$, since $17 = 17 + (x - c)0$.

The identity function $x \mapsto x$ has derivative $c \mapsto 1$, since $x = c + (x - c)1$.

In general, for any positive integer n, the function $x \mapsto x^n$ has derivative $c \mapsto nc^{n-1}$, since $x^n = c^n + (x - c)(x^{n-1} + cx^{n-2} + \cdots + c^{n-1})$.

The function $x \mapsto 1/x$, defined for nonzero inputs, has derivative $c \mapsto -1/c^2$. Indeed, $1/x = 1/c + (x - c)(-1/cx)$, and $x \mapsto -1/cx$ is continuous at c with value $-1/c^2$.

Now the chain rule, with $f = (x \mapsto 1/x)$, states that $1/g$ has derivative $-g'/g^2$ at any point c where $g(c) \neq 0$. In particular, for any negative integer n, $x \mapsto x^n$ has derivative $c \mapsto nc^{n-1}$.

Finally, the product rule implies that h/g has derivative $(gh' - hg')/g^2$ at any point c where $g(c) \neq 0$; this is the **quotient rule**.

Part III. Completeness and its consequences

8. Completeness of the real numbers

Definition 8.1. *Let S be a set of real numbers. A real number c is an* **upper bound for** *S if $x \leq c$ for all $x \in S$.*

For example, any number $c \geq \pi$ is an upper bound for the set $\{3, 3.1, 3.14, 3.141, \ldots\}$. The smallest upper bound is π.

The real numbers are **complete**: if S is a nonempty set, and there is an upper bound for S, then there is a smallest upper bound for S. The smallest upper bound is unique; it is called the **supremum of** S, written $\sup S$.

9. The intermediate-value theorem

Theorem 9.1. *Let f be a continuous real-valued function. Let y be a real number. Let $b \le c$ be real numbers with $f(b) \le y \le f(c)$. Then $f(x) = y$ for some $x \in [b, c]$.*

Here $[b, c]$ means $\{x : b \le x \le c\}$. For example, if $f(3) = -5$ and $f(4) = 7$, and f is continuous, then f must have a root between 3 and 4.

Proof. Define $S = \{x \in [b, c] : f(x) \le y\}$. S is nonempty, because it contains b, and it has an upper bound, namely c, so it has a smallest upper bound, say u.

Suppose $f(u) > y$. By continuity, there is an open ball D around u such that $f(x) > y$ for $x \in D$. Pick any $t \in D$ with $t < u$. If $x \in [t, u]$ then $x \in D$ so $f(x) > y$ so $x \notin S$. Thus t is an upper bound for S—but u is the smallest upper bound. Contradiction.

Suppose $f(u) < y$. Then $u \ne c$ so $u < c$. By continuity, there is an open ball D around u such that $f(x) < y$ for $x \in D$. Pick any $x \in D$ with $u < x < c$; then $f(x) < y$. But $x \notin S$ since u is an upper bound for S; so $f(x) > y$. Contradiction. □

10. The maximum-value theorem

Theorem 10.1. *Let f be a continuous real-valued function. Let $b \le c$ be real numbers. Then there is an upper bound for $f([b, c])$.*

Proof. Let S be the set of $x \in [b, c]$ such that $f([b, x])$ is bounded—i.e., has an upper bound. S is nonempty, because it contains b. Define $u = \sup S$.

By continuity, there is an open ball D around u such that $f(D) \subseteq \text{Ball}(f(u), 1)$. Select $t \in D$ with $t < u$; then t is not an upper bound for S, so there is some $x \in S$ with $t < x \le u$. Now $f([b, x])$ and $f([x, u]) \subseteq f(D)$ are bounded, so $f([b, u])$ is bounded.

Suppose $u < c$. Select $v \in D$ with $u < v < c$. Then $f([u, v])$ is bounded, so $v \in S$. Contradiction. Hence $u = c$, and $f([b, c]) = f([b, u])$ is bounded. □

Theorem 10.2. *Let f be a continuous real-valued function. Let $b \le c$ be real numbers. Then there is some $u \in [b, c]$ such that, for all $z \in [b, c]$, $f(u) \ge f(z)$.*

This is the **maximum-value theorem**: a continuous function on a closed interval achieves a maximum. The same is not true for open intervals: consider $1/x$ for $0 < x < 1$.

Proof. By Theorem 10.1, there is an upper bound for $f([b, c])$. Define $M = \sup f([b, c])$.

Let S be the set of $x \in [b, c]$ such that $\sup f([x, c]) = M$. Then $b \in S$. Define $u = \sup S$.

Suppose $f(u) < M$. By continuity there is an open ball D around u such that $f(D) \subseteq \text{Ball}(f(u), (M - f(u))/2)$; then $\sup f(D) < M$. Select $t \in D$ with $t < u$; then t is not an upper bound for S, so there is some $x \in S$ with $t < x \le u$. Then $\sup f([x, c]) = M$, but $\sup f([x, u]) < M$, so $u < c$. Select $v \in D$ with $u < v < c$. Then $\sup f([x, v]) < M$, so $\sup f([v, c]) = M$, so $v \in S$. Contradiction. Hence $f(u) = M = \sup f([b, c])$. □

Theorem 10.3. *Let f be a continuous real-valued function. Let $b \leq c$ be real numbers. Then there is some $u \in [b, c]$ such that, for all $x \in [b, c]$, $f(u) \leq f(x)$.*

Proof. Apply Theorem 10.2 to $-f$. □

Part IV. The mean-value theorem

11. Fermat's principle

Theorem 11.1. *Let f be a real-valued function differentiable at t. Assume that $f(t) \geq f(x)$ for all x in an open ball B around t. Then $f'(t) = 0$.*

Proof. By assumption $f(x) = f(t) + (x - t)f_1(x)$ where f_1 is continuous at t. Suppose $f_1(t) > 0$. Then $f_1(x) > 0$ for all x in an open ball D around t. Pick $x > t$ in both B and D; then $f(t) \geq f(x) = f(t) + (x - t)f_1(x) > f(t)$. Contradiction. Thus $f_1(t) \leq 0$. Similarly $f_1(t) \geq 0$. Hence $f'(t) = f_1(t) = 0$. □

Theorem 11.2. *Let f be a real-valued function differentiable at t. Assume that $f(t) \leq f(x)$ for all x in an open ball B around t. Then $f'(t) = 0$.*

Proof. Apply Theorem 11.1 to $-f$. □

12. Rolle's theorem

Theorem 12.1. *Let f be a differentiable real-valued function. Let $b < c$ be real numbers. If $f(b) = f(c)$ then there is some x with $b < x < c$ such that $f'(x) = 0$.*

Proof. By Theorem 10.2, there is some $t \in [b, c]$ such that f's maximum value on $[b, c]$ is achieved at t. If $f(t) > f(b)$ then $t \neq b$ and $t \neq c$, so there is an open ball B around t such that $B \subseteq [b, c]$. By Theorem 11.1, $f'(t) = 0$.

Similarly, by Theorem 10.3, there is some $u \in [b, c]$ such that f achieves its minimum at u. If $f(u) < f(b)$ then $f'(u) = 0$ as above.

The only remaining case is that $f(t) \leq f(b)$ and $f(u) \geq f(b)$. Then $f(b)$ is both the maximum and the minimum value of f on $[b, c]$; i.e., f is constant on $[b, c]$. Hence $f'(x) = 0$ for any x between b and c. □

13. The mean-value theorem

Theorem 13.1. *Let f be a differentiable real-valued function. Let $b < c$ be real numbers. Then there is some x with $b < x < c$ such that $f(c) - f(b) = f'(x)(c - b)$.*

This is the **mean-value theorem**. The terminology "mean value" comes from the fundamental theorem of calculus, which can be interpreted as saying that $(f(c)-f(b))/(c-b)$ is the average ("mean") value of $f'(x)$ for $x \in [b,c]$. See Theorem 16.1.

Proof. Define $g(x) = (c-b)f(x) - (x-b)(f(c) - f(b))$. Then g is differentiable, and $g(b) = (c-b)f(b) = (c-b)f(c) - (c-b)(f(c) - f(b)) = g(c)$. By Theorem 12.1, $g'(x) = 0$ for some x between b and c. But $g'(x) = (c-b)f'(x) - (f(c) - f(b))$. □

Theorem 13.2. *Let f be a differentiable real-valued function. If $f'(x) = 0$ for all x then f is constant.*

More generally, two functions with the same derivative must differ by a constant.

Proof. Pick any real numbers $b < c$. By Theorem 13.1, there is some x such that $f(c) - f(b) = f'(x)(c-b) = 0$, so $f(c) = f(b)$. □

Part V. Integration

14. Tagged divisions and gauges

Definition 14.1. *Let $b \le c$ be real numbers. Let $x_0, x_1, \ldots, x_n$ and $t_1, \ldots, t_n$ be real numbers. Then $x_0, t_1, x_1, \ldots, t_n, x_n$ is a* **tagged division of** $[b,c]$ *if $b = x_0 \le t_1 \le x_1 \le t_2 \le \cdots \le x_{n-1} \le t_n \le x_n = c$.*

The idea is that $[b,c]$ is divided into the intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$; in each interval $[x_{k-1}, x_k]$ there is a *tag* t_k. For example, consider the tagged division $0, 1, 4, 5, 6, 6, 7$ of $[0,7]$; here the intervals are $[0,4], [4,6], [6,7]$, with tags $1, 5, 6$ respectively.

Definition 14.2. *Let $b \le c$ be real numbers. A* **gauge on** $[b,c]$ *is a function assigning to each point $t \in [b,c]$ an open interval containing t.*

For example, given $\epsilon > 0$, the function $t \mapsto \text{Ball}(t, \epsilon)$ is a gauge on any interval.

Definition 14.3. *Let $b \le c$ be real numbers. Let G be a gauge on $[b,c]$. A tagged division $x_0, t_1, x_1, \ldots, t_n, x_n$ of $[b,c]$ is* **inside** G *if $[x_{k-1}, x_k] \subset G(t_k)$ for every k.*

Theorem 14.4. *Let $b \le c$ be real numbers. Let G be a gauge on $[b,c]$. Then there is a tagged division of $[b,c]$ inside G.*

Proof. Let S be the set of $x \in [b,c]$ such that there is a tagged division of $[b,x]$ inside G. S is nonempty: b, b, b is a tagged division of $[b,b]$ inside G, so $b \in S$. Also, c is an upper bound for S. Thus there is a smallest upper bound for S, say y.

Select $v \in G(y)$ such that $v < y$. Then v is not an upper bound for S, so there is some $x > v$ with $x \in S$. Let $x_0, t_1, x_1, \ldots, t_n, x_n$ be a tagged division of $[b,x]$ inside G.

Suppose $y < c$. Pick $z \in G(y)$ with $y < z \leq c$. Then $[x_n, z] = [x, z] \subset [v, z] \subset G(y)$, so $x_0, t_1, x_1, \ldots, t_n, x_n, y, z$ is a tagged division of $[b, z]$ inside G. Thus $z \in S$; but $z > y$, and y is an upper bound for S. Contradiction.

Thus $y = c$. Finally $x_0, t_1, x_1, \ldots, t_n, x_n, y, y$ is a tagged division of $[b, c]$ inside G. □

15. The definite integral

Definition 15.1. *Let $b \leq c$ be real numbers. Let $x_0, t_1, x_1, \ldots, t_n, x_n$ be a tagged division of $[b, c]$. Let f be a function defined on $[b, c]$. The* **Riemann sum for f on** *$x_0, t_1, x_1, \ldots, t_n, x_n$ is $(x_1 - x_0)f(t_1) + \cdots + (x_n - x_{n-1})f(t_n)$.*

For example, the Riemann sum for f on $0, 1, 4, 5, 6, 6, 7$ is $(4-0)f(1) + (6-4)f(5) + (7-6)f(6)$. This may be visualized as the sum of areas of three rectangles: one stretching from 0 to 4 horizontally with height $f(1)$, another from 4 to 6 with height $f(5)$, and another from 6 to 7 with height $f(6)$.

Definition 15.2. *Let $b \leq c$ be real numbers. Let f be a function defined on $[b, c]$. Let I be a number. Then f* **has integral I on** *$[b, c]$ if, for every open ball E around I, there is a gauge G on $[b, c]$ such that E contains the Riemann sum for f on any tagged division of $[b, c]$ inside G.*

Theorem 15.3. *Let $b \leq c$ be real numbers. Let f be a function. If f has integral I on $[b, c]$ and f has integral J on $[b, c]$ then $I = J$.*

Thus there is at most one number I such that f has integral I on $[b, c]$. If this number exists, it is called the **integral of f from b to c**, written $\int_b^c f$.

Proof. I will show that $|I - J| < 2\epsilon$ for any $\epsilon > 0$.

By definition of integral, there is a gauge G on $[b, c]$ such that $\text{Ball}(I, \epsilon)$ contains the Riemann sum for f on any tagged division of $[b, c]$ inside G.

Similarly, there is a gauge H on $[b, c]$ such that $\text{Ball}(J, \epsilon)$ contains the Riemann sum for f on any tagged division of $[b, c]$ inside G.

Define $F(t)$ as the intersection of $G(t)$ and $H(t)$. Then F is a gauge on $[b, c]$. By Theorem 14.4, there is a tagged division $x_0, \ldots, x_n$ of $[b, c]$ inside F.

Let R be the Riemann sum for f on $x_0, \ldots, x_n$. Observe that $x_0, \ldots, x_n$ is inside both G and H, so $R \in \text{Ball}(I, \epsilon)$ and $R \in \text{Ball}(J, \epsilon)$. Hence $|I - J| \leq |I - R| + |R - J| < 2\epsilon$. □

16. The fundamental theorem of calculus

Theorem 16.1. *Let f be a differentiable function. Let $b \leq c$ be real numbers. Then $f(c) - f(b) = \int_b^c f'$.*

Proof. Pick $\epsilon > 0$. I will construct a gauge G such that $\text{Ball}(f(c) - f(b), \epsilon(c - b + 1))$ contains the Riemann sum for f' on any tagged division of $[b, c]$ inside G.

Fix $t \in [b, c]$. Since f is differentiable at t, there is a function f_1, continuous at t, such that $f(x) = f(t) + (x - t)f_1(x)$. By definition of continuity, $f_1(x)$ is within ϵ of $f_1(t) = f'(t)$ for all x in some open ball around t. Define $G(t)$ as the union of all such balls. Then G is a gauge on $[b, c]$.

Observe that if $x, y \in G(t)$, with $x \le t \le y$, then $(y - x)f'(t)$ is within $\epsilon(y - x)$ of $f(y) - f(x)$. Indeed, $|f_1(x) - f'(t)| < \epsilon$ by definition of G, and $f(x) - f(t) = (x - t)f_1(x)$, so

$$|f(x) - f(t) - (x - t)f'(t)| = |(x - t)(f_1(x) - f'(t))| \le \epsilon\, |x - t|\,.$$

Similarly $|f(y) - f(t) - (y - t)f'(t)| \le \epsilon\, |y - t|$. Thus $|f(y) - f(x) - (y - x)f'(t)| \le \epsilon(|y - t| + |x - t|)$; and $|y - t| + |x - t| = y - x$.

Finally, say $x_0, t_1, x_1, \ldots, t_n, x_n$ is a tagged division of $[b, c]$ inside G. Then $x_{k-1}, x_k \in G(t_k)$, with $x_{k-1} \le t_k \le x_k$, so $(x_k - x_{k-1})f'(t_k)$ is within $\epsilon(x_k - x_{k-1})$ of $f(x_k) - f(x_{k-1})$ as above. Thus the Riemann sum for f' on $x_0, t_1, x_1, \ldots, t_n, x_n$ is within

$$\sum_{1 \le k \le n} \epsilon(x_k - x_{k-1}) = \epsilon(x_n - x_0) = \epsilon(c - b) < \epsilon(c - b + 1)$$

of

$$\sum_{1 \le k \le n} (f(x_k) - f(x_{k-1})) = f(x_n) - f(x_0) = f(c) - f(b)$$

as claimed. □

17. Integration rules

Theorem 17.1. *Let f be a function. Let $b \le c$ be real numbers. If $\int_b^c f = I$ then af has integral aI on $[b, c]$ for any real number a.*

In short $\int_b^c af = a \int_b^c f$ if the right side is defined.

Proof. Pick $\epsilon > 0$. Since $\int_b^c f = I$, there is a gauge G on $[b, c]$ such that $\text{Ball}(I, \epsilon)$ contains the Riemann sum for f on any tagged division of $[b, c]$ inside G. The Riemann sum for af is exactly a times the Riemann sum for f, so it is inside $\text{Ball}(aI, |a|\, \epsilon)$ for $a \ne 0$ or $\text{Ball}(0, \epsilon)$ for $a = 0$. □

Theorem 17.2. *Let f and g be functions. Let $b \le c$ be real numbers. If $\int_b^c f = I$ and $\int_b^c g = J$ then $f + g$ has integral $I + J$ on $[b, c]$.*

In short $\int_b^c (f + g) = \int_b^c f + \int_b^c g$ if the right side is defined.

Proof. Pick $\epsilon > 0$. There is a gauge F on $[b, c]$ such that $\text{Ball}(I, \epsilon)$ contains the Riemann sum for f on any tagged division of $[b, c]$ inside F; and there is a gauge G on $[b, c]$ such that $\text{Ball}(J, \epsilon)$ contains the Riemann sum for g on any tagged division of $[b, c]$ inside G.

Define $H(t) = F(t) \cap G(t)$. Then H is a gauge on $[b, c]$. If $x_0, \ldots, x_n$ is a tagged division of $[b, c]$ inside H, then $x_0, \ldots, x_n$ is also inside both F and G, so the Riemann sums for f and g on $x_0, \ldots, x_n$ are within ϵ of I and J respectively; thus the Riemann sum for $f + g$ on $x_0, \ldots, x_n$ is within 2ϵ of $I + J$. □

Theorem 17.3. *Let f be a function. Let $a \le b \le c$ be real numbers. If $\int_a^b f = I$ and $\int_b^c f = J$ then f has integral $I + J$ on $[a, c]$.*

In short $\int_a^c f = \int_a^b f + \int_b^c f$ if the right side is defined.

Proof. Pick $\epsilon > 0$. There is a gauge G on $[a, b]$ such that $\text{Ball}(I, \epsilon)$ contains the Riemann sum for f on any tagged division of $[a, b]$ inside G; there is a gauge H on $[b, c]$ such that $\text{Ball}(J, \epsilon)$ contains the Riemann sum for f on any tagged division of $[b, c]$ inside H.

I define a new gauge as follows. For $t < b$ define $F(t) = \{x \in G(t) : x < b\}$. For $t = b$ define $F(t) = G(t) \cap H(t)$. For $t > b$ define $F(t) = \{x \in H(t) : x > b\}$.

Say $x_0, \ldots, x_n$ is a tagged division of $[a, c]$ inside F. Then $b \in [x_{k-1}, x_k] \subset F(t_k)$ for some k; by construction of F, t_k must equal b. Now $x_0, t_1, x_1, \ldots, x_{k-1}, t_k, b$ is a tagged division of $[a, b]$ inside F, hence inside G. Thus the Riemann sum $(x_1 - x_0)f(t_0) + \cdots + (b - x_{k-1})f(t_k)$ is within ϵ of I. Similarly the Riemann sum $(x_k - b)f(t_k) + \cdots + (x_n - x_{n-1})f(t_n)$ is within ϵ of J. Add: the Riemann sum $(x_1 - x_0)f(t_0) + \cdots + (x_k - x_{k-1}f(t_k) + \cdots + (x_n - x_{n-1})f(t_n)$ is within 2ϵ of $I + J$. □

Theorem 17.4. *Let f be a function. Let $b \le c$ be real numbers. If f is nonnegative on $[b, c]$ and $\int_b^c f = I$ then I is nonnegative.*

Proof. Pick $\epsilon > 0$. Select an appropriate gauge G. By Theorem 14.4, there is an appropriate tagged division of $[b, c]$. The corresponding Riemann sum is nonnegative, so $I \ge -\epsilon$. □

Part VI. Limits

18. Convergence and limits

Definition 18.1. *Let f be a function. Then f* **converges to** L **at** c *if the function*

$$x \mapsto \begin{cases} L & \text{if } x = c \\ f(x) & \text{if } x \ne c \end{cases}$$

is continuous at c.

Equivalent terminology: $f(x)$ **converges to** L **as** x **approaches** c.

By Theorem 2.2, there is at most one number L such that f converges to L at c. If this number exists, it is called **the limit of** f **at** c, or **the limit of** $f(x)$ **as** x **approaches** c, written $\lim_{x \to c} f(x)$. Note that f is continuous if and only if $\lim_{x \to c} f(x) = f(c)$.

Example: $\cos(1/x)$ does not converge to 0 as x approaches 0.

19. Limits of sums, products, and compositions

Theorem 19.1. *Let f and g be functions. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ then $f(x) + g(x)$ converges to $L + M$ as x approaches c.*

In short $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$ if the right side is defined.

Proof. Replace $f(c)$ by L and $g(c)$ by M to obtain new functions a and b. Then a and b are continuous, so $a + b$ is continuous by Theorem 3.1. □

Theorem 19.2. *Let f and g be functions. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ then $f(x)g(x)$ converges to LM as x approaches c.*

Proof. Theorem 3.2. □

Theorem 19.3. *Let f and g be functions. If $\lim_{x\to c} g(x) = L$, and f is continuous at L, then $f(g(x))$ converges to $f(L)$ as x approaches c.*

In short $\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$ if the right side is defined, provided that f is continuous.

Proof. Theorem 3.3. □

20. L'Hôpital's rule

Theorem 20.1. *Let f and g be real-valued functions differentiable at c. If $f(c) = g(c) = 0$, and $g'(c) \neq 0$, then $f(x)/g(x)$ converges to $f'(c)/g'(c)$ as x approaches c.*

For example, $\lim_{x\to 0}(x/\sin x) = 1/1 = 1$, since $\sin' = \cos$ and $\cos 0 = 1 \neq 0$.

Proof. By assumption $f(x) = f(c) + (x - c)f_1(x) = (x - c)f_1(x)$ where f_1 is continuous at c. Similarly $g(x) = (x - c)g_1(x)$ where g_1 is continuous at c. By assumption $g_1(c) = g'(c) \neq 0$, so the function $x \mapsto f_1(x)/g_1(x)$ is continuous at c, with value $f_1(c)/g_1(c)$. Finally $f(x)/g(x) = f_1(x)/g_1(x)$ for $x \neq c$. □

Theorem 20.2. *Let f and g be differentiable real-valued functions. If $f(c) = g(c) = 0$, and $\lim_{x\to c}(f'(x)/g'(x)) = L$, then $f(x)/g(x)$ converges to L as x approaches c.*

Proof. Fix a ball E around L. There is a ball D around c such that $f'(x)/g'(x) \in E$ for all $x \in D$ with $x \neq c$. In particular, $g'(x)$ is nonzero for $x \in D$. By Theorem 12.1, $g(y)$ is nonzero for $y \in D$.

I will show that $f(y)/g(y) \in E$ for all $y \in D$ with $y \neq c$. Thus $f(y)/g(y)$ converges to L as y approaches c.

Given $y \in D$, $y \neq c$, consider the function $h = (x \mapsto f(x)g(y) - f(y)g(x))$. Notice that h is differentiable, with $h'(x) = f'(x)g(y) - f(y)g'(x)$.

Now $h(c) = f(c)g(y) - f(y)g(c) = 0$, and $h(y) = f(y)g(y) - f(y)g(y) = 0$, so there is some x between c and y with $h'(x) = 0$ by Theorem 12.1. Thus $f'(x)g(y) = f(y)g'(x)$. Both $g'(x)$ and $g(y)$ are nonzero, so $f(y)/g(y) = f'(x)/g'(x) \in E$. □

Theorem 20.2 may be used repeatedly. For example:

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

99. Expository notes

Common practice in calculus books is to define continuity using limits. I define limits using continuity; continuity is a simpler concept.

"An open ball around c" is substantially easier to read than "for some $h > 0$, the set of x such that $|x - c| < h$."

I use Carathéodory's definition of the derivative of f. The point is to give a name to the function $x \mapsto (f(x) - f(c))/(x - c)$. I learned about this from an article by Stephen Kuhn in the *Monthly*. It's also used in the second edition of Apostol's text.

My proof of Theorem 6.2 uses the formula $h_1(x) = f_1(x)g(x) + f(c)g_1(x)$, which is shorter than the (more obvious) formula $h_1(x) = f_1(x)g(c) + f(c)g_1(x) + (x - c)f_1(x)g_1(x)$. I was reminded of this simplification by a letter in the *Monthly* from Günter Pickert.

The Heine-Borel theorem follows immediately from Theorem 14.4. See Botsko's 1987 *Monthly* article for this approach to all the basic completeness theorems. Thanks to Joe Buhler for the reference.

I follow the Kurzweil-Henstock approach to integration. The resulting integral is *more* general than the Lebesgue integral; it is equivalent to the integrals constructed by Denjoy and Perron. There is no need for any technical conditions in the fundamental theorem of calculus, Theorem 16.1; every derivative is integrable. I learned about this from advertisements by Robert G. Bartle in the *Bulletin* and the *Monthly*.