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RICCI FLOW WITH SURGERY

Diplomarbeit
Betreuer: Bernhard Leeb

eingereicht von Richard H Bamler
am 28. August 2007

Erklärung

Hiermit erkläre ich, Richard Heiner Bamler, dass ich die vorliegende Diplomarbeit mit dem Titel

"Ricci flow with surgery"

selbständig verfasst und nur die angegebenen Quellen und Hilfsmittel benutzt habe.

München, August 27, 2007

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Introduction

The following thesis deals with the surgery process for the Ricci flow on 3-manifolds as developed in [Per1] and [Per2]. With the help of this method it is possible to prove the Poincaré Conjecture as well as the Geometrization Conjecture.

Poincaré Conjecture. *Every closed (i.e. compact and boundaryless) simply connected* 3*-manifold M is homeomorphic to S^3.*

Note that any topological 3-manifold admits a unique differentiable structure, so it remains to show that every closed simply connected smooth 3-manifold is homeomorphic to S^3.

The Geometrization Conjecture reads as follows:

Geometrization Conjecture. *Every closed orientable* 3*-manifold M is a connected sum of closed* 3*-manifolds M_i such that for every i there is a finite collection of pairwise disjoint imbedded tori $T_{ij} \subset M_i$ such that*

(i) the tori $\{T_{ij}\}_j$ are incompressible in M_i (i.e. $\pi_1 T_{ij} \hookrightarrow \pi_1 M_i$)

(ii) the components of $M_i \setminus \bigcup_j T_{ij}$ are diffeomorphic to metric quotients with finite volume of one of the following 8 homogeneous geometries

$$S^3, \ S^2 \times S^1, \ \mathbb{H}^3, \ \mathbb{R}^3, \ \mathbb{H}^2 \times \mathbb{R}, \ \widetilde{\mathrm{PSL}}(2, \mathbb{R}), \ \mathrm{Nil} \qquad \textit{or} \qquad \mathrm{Sol}.$$

For more details see [Mor], [Thu] or [KL, Appx I]. Note that the Poincaré Conjecture follows easily from the Geometrization Conjecture.

The idea for proving the Poincaré conjecture and the Geometrization Conjecture for a manifold M via Ricci flow is the following: We endow M with an arbitrary metric and evolve it via the Ricci flow. Naïvely we can expect the metric to get more and more homogeneous since the evolution equation for the curvature under the Ricci flow is of heat type. So for example in the case in which the starting metric has positive Ricci curvature, Richard Hamilton showed in [Ham1] that the metric converges to a constant curvature metric under the Ricci flow. However, in the general case it may happen that the metric develops regions of diverging curvature called *singularities.* Perelman has found ways (see [Per1]) to prove that the injectivity radius in these regions does not decrease faster than the curvature diverges. This gives him the possibility to conclude that the singularities are geometrically close to a certain type of ancient Ricci flows called *κ-solutions.* It is now an important step to analyze these model solutions and to find they look neck-like at certain areas, that is, their geometry is close to the homogenoeous geometry $S^2 \times \mathbb{R}$. This in turn implies that the singularities essentially look neck-like, too. In order to get rid of the singularities, one can cut out those necks and glue 3-balls whose geometry should be chosen in a certain way into the produced cutting surfaces. This process is called *surgery.* Now one can start the Ricci flow again and repeat the surgery procedure if necessary. Note that topologically the surgery process corresponds to performing the inverse of a connected sum.

This thesis is organized as follows: Chapters 1 and 2 give a very brief introduction to results in Riemannian geometry and Ricci flow that will be needed subsequently. Most often proofs are omitted if there are appropriate references. In chapter 3 we present an overview on geometric convergence and geometric compactness theorems. Results on Gromov-Hausdorff compactness are assumed to be familiar to the reader and will only be repeated without proofs. However, the theory of smooth convergence and convergence of Ricci flows is presented more thoroughly since one hardly finds a complete and accurate exposition on the topic in the literature. The proof of only one technical fact is omitted, but reference is given. Amongst others we try to convey a certain languague to the reader in this chapter that will be used in the following chapters. Chapter 4 gives a short introduction to the methods used to prove Perelman's No Local Collapsing Theorem. These methods will be used in chapter 7 again. References for more detailed presentations of the topics are given. In chapter 5 the theory of κ-solution is developed and chapter 6 presents the Canonical Neighborhood Theorem (compare with 12.1 in [Per1]). Chapters 5 and 6 follow Bruce Kleiner's and John Lott's notes on Perelman's papers [KL] with some modifications. All proofs are given except for the proof of the theorem that there is a universal κ_0 for all non-round 3-dimensional κ-solutions. Note that in chapter 6 we try to cover as complete as possible the results on the classification of 3 dimensional κ-solutions. However, some of these results will not be needed in the following chapters. Chapter 7 deals with the surgery process in the Ricci flow. At first, general Ricci flows with surgery are introduced. Then we give a brief description of the surgery process. Some technical details are omitted, but can be found in [KL]. Eventually, in the last three sections we prove that it is always possible to perform the surgeries in the described way. Finally, chapter 8 gives a brief overview on the methods that can now be used to prove the Poincaré or the Geometrization Conjecture.

At this point we like to list other sources that cover Ricci flow with surgery. Very useful are [KL], [MT] and [Hei1]. Sometimes, Perelman's original papers [Per1] and [Per2] are quite helpful although they present the material in a very condensed way. Another reference is [CZ]. For a detailed overview on the topic see [Mor].

The author is aware of the fact that the references [Bam1], [Bam2], [LB1], [LB2], [LB3], [LB4] and [Lee2] are hardly accessible to the reader. Hopefully, these sources will be added to a future version of this text.

This diploma thesis was written at the Ludwig-Maximilians University, Munich. I like to thank my advisor Bernhard Leeb for the numerous inspirations and help. Furthermore, I like to thank Hans-Joachim Hein for his support and intensive reading circles.

Chapter 1

Preliminaries on geometry and topology

1.1 Notations and conventions

Let (M, g) be an n dimensional Riemannian manifold, i.e. a smooth manifold with a symmetric and positive definite bilinear form $g \in \mathrm{Sym}^2 T^*M$, the *metric.* In a canonical way, the metric g induces metrics on all higher tensor bundles $T_l^k TM = (TM)^{\otimes l} \otimes (T^*M)^{\otimes k}$ and their subbundles resp. quotients such as $\Lambda^k TM$ or $\mathrm{Sym}^k TM$. If M is orientable, let $\mathrm{vol} \in \Gamma\Lambda^n TM = \Omega^n M$ be the *Riemannian volume form,* i.e. the form that satisfies $\mathrm{vol}(e_1, \ldots, e_n) = 1$ for any oriented local orthormal frame (e_i). Define the *Riemannian measure* μ to be the measure on M associated to vol. Observe that working locally, we can also define μ in the case in which M is non-orientable.

We denote by ∇ the *Levi-Civita connection* on TM, i.e. the connection that satisfies $\nabla_X Y - \nabla_Y X = [X, Y]$ (i.e. ∇ is *torsion free*) and $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ (i.e. ∇ is a *metric* connection) for all vector fields X, Y, Z on M. Analogously, ∇ induces metric connections on all higher tensor bundles $T_l^k TM$ and their parallel subbundles or quotients. Let E be a vector bundle over M equipped with a connection ∇. We define the *curvature tensor* $R \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}\, E)$ by

$$R(X, Y)e := \nabla_X \nabla_Y e - \nabla_Y \nabla_X e - \nabla_{[X,Y]} e$$

for any two vector fields $X, Y \in \Gamma(TM)$ and section $e \in \Gamma(E)$. It is easy to see that R is indeed a tensor. In the case $E = TM$, the curvature R is the *Riemannian curvature.* Let $p \in M$ and $\pi \subset T_pM$ be a 2-plane. Choose a basis $u, v \in \pi$ and define the *sectional curvature* on π by

$$K(\pi) = K(u \wedge v) := \frac{\langle R(u, v)v, u\rangle}{\|u \wedge v\|^2} \geq \kappa \qquad \text{where} \qquad \|u \wedge v\|^2 = \|u\| \cdot \|v\| - \langle u, v\rangle^2.$$

One easily checks that $K(u \wedge v)$ does not depend on the choice of $u, v \in \pi$. We say that M has *constant sectional curvature* κ if $K(\pi) \equiv \kappa$ for all points $p \in M$ and 2-planes $\pi \subset T_pM$. For each $n \geq 2$ and $\kappa \in \mathbb{R}$ there is up to isometry exactly one complete and simply connected Riemannian manifold with constant sectional curvature κ. We will call this space the *model space of constant sectional curvature κ, M_κ^n.*

Finally we define the *Riemannian curvature operator* $\hat{R} \in \Gamma(\mathrm{End}^{\mathrm{sym}} \Lambda^2 TM)$ by

$$\langle \hat{R}(X \wedge Y), U \wedge V\rangle = \langle R(X, Y)V, U\rangle$$

for any vector fields X, Y, U, V (note again that we set $\langle X \wedge Y, U \wedge V\rangle := \langle X, U\rangle\langle Y, V\rangle - \langle X, V\rangle\langle Y, U\rangle$).

Contracting the Riemannian curvature yields the *Ricci curvature:*

$$\operatorname{Ric}(X,Y) := \operatorname{tr} R(\cdot, X)Y = \sum_i \langle R(X,e_i)e_i, Y\rangle$$

for any local vector fields X, Y and any local orthonormal frame (e_i). Note that $\operatorname{Ric} \in \Gamma(\operatorname{Sym}^2 T^*M)$, however sometimes we dualize and assume that $\operatorname{Ric} \in \Gamma\left(\operatorname{End}^{\mathrm{sym}} TM\right)$.

Tracing the Ricci tensor gives the *scalar curvature:*

$$S = \operatorname{tr}\operatorname{Ric} = \sum_i \operatorname{Ric}(e_i, e_i).$$

For a more developed introduction to Riemannian curvature quantities see [dCa].

1.2 Metric geometry

In the following denote (X, d_X) a metric space. If there is no chance for confusion, we will simply write d instead of d_X for the metric.

Definition 1.2.1. *A metric space (X,d) is called a* length space *if for all $x, y \in X$ we have*

$$d(x,y) = \inf\left\{\ell(\gamma) \quad : \quad \begin{array}{c} \gamma : [0,1] \to X \ \text{ continuous} \\ \gamma(0) = x, \ \ \gamma(1) = y \end{array}\right\}$$

where $\ell(\gamma)$ denotes the length *of γ, i.e.*

$$\ell(\gamma) := \sup\left\{\sum_{k=1}^{n} d(\gamma(t_{k-1}), \gamma(t_k)) \quad : \quad n \in \mathbb{N}, \ \ 0 = t_0 < t_1 < \ldots < t_n = 1\right\}$$

For a Riemannian manifold (M, g) we define the *path metric* dist by

$$\operatorname{dist}(x,y) := \inf\left\{\int_0^1 \|\dot\gamma\| \mathrm{d}t \quad : \quad \gamma : [0,1] \to M, \ \ \mathcal{C}^1, \ \ \gamma(0) = x, \gamma(1) = y\right\}$$

Obviously, (M, dist) is a length space. In order to make sure that the property of being a length space persists Gromov-Hausdorff limits we mention (see [Bal1])

Lemma 1.2.2. *For a metric space (X,d) the following properties are equivalent:*
(i) X is a length space
(ii) For any $\varepsilon > 0$ and any two points $x, y \in X$ there is a point $z \in X$ such that

$$d(x,z) - \tfrac{1}{2}d(x,y), \ \ d(z,y) - \tfrac{1}{2}d(x,y) < \varepsilon$$

Definition 1.2.3. *Let $I \subset \mathbb{R}$ be an interval equipped with the standard metric d_I. A map $\gamma : I \to X$ is called a* minimizing geodesic *if γ is an isometry between (I, cd_I) and (X, d_X) for some $c \geq 0$. If $c = 1$ we say that γ is a* minimizing geodesic parameterized by arclength. *If $I = \mathbb{R}_{\geq 0}$, γ will be called a* ray *and if $I = \mathbb{R}$, a* line.

We call γ a geodesic (parameterized by arclength) *if for any $t \in I$ there is an interval $J \subset I$ around t such that $\gamma|_J$ is a minimizing geodesic (parameterized by arclength).*

Under certain conditions, we are able to formulate a synthetic version of the Hopf-Rinow Theorem in the Riemannian case (see [dCa, Ch 7]). For the proof see [Bal1, I.2]

Proposition 1.2.4. *Let (X,d) be a locally compact length space. Then the following conditions are equivalent:*
(a) X is complete as a metric space,
(b) any geodesic $\gamma : [0,1) \to X$ can be extended to a geodesic on $[0,1]$,

(c) *for some* $x \in X$*, any geodesic* $\gamma : [0,1) \to X$ *starting in* x*, i.e.* $\gamma(0) = x$*, can be extended to a geodesic on* $[0,1]$*,*

(d) *bounded subsets are precompact (i.e. their closure is compact).*

Furthermore, (a)-(d) imply

(e) *for any pair of points* $x, y \in X$ *there is a minimizing geodesic* $\gamma : [0,1] \to X$ *with* $\gamma(0) = x$ *and* $\gamma(1) = y$*.*

1.3 Comparison geometry

Let (M, g) be a Riemannian manifold and $x_1, x_2, x_3 \in M$ be points joined by (not necessarily minimizing) geodesics $\gamma_1, \gamma_2, \gamma_3 \subset M$ in such a way that γ_i connects x_{i-1} with x_{i+1} (where we view indices always modulo 3). We call the triple $(\gamma_1, \gamma_2, \gamma_3)$ a *triangle* in M if $\ell(\gamma_i) + \ell(\gamma_{i+1}) \geq \ell(\gamma_{i+2})$ for all $i = 1, 2, 3$. If there is no chance of confusion, we will abbreviate $\triangle x_1x_2x_3 := (\gamma_1, \gamma_2, \gamma_3)$ and denote the sides γ_i by $\overline{x_{i-1}x_{i+1}}$.

Let $\kappa \in \mathbb{R}$. If the perimeter $\ell(\gamma_1) + \ell(\gamma_2) + \ell(\gamma_3) \leq \frac{2\pi}{\sqrt{\kappa}}$ (we set $\frac{2\pi}{\sqrt{\kappa}} = \infty$ if $\kappa \leq 0$), there exists a triangle $\triangle\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3 = (\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3)$ in the model space M^2_κ of constant curvature κ with $\ell(\gamma_i) = \ell(\widetilde{\gamma}_i)$. Note that the length of no side exceeds $\frac{\pi}{\sqrt{\kappa}}$. We call $\triangle\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3$ a *comparison triangle* of $\triangle x_1x_2x_3$. In the case in which all sides of the triangle are smaller than $\frac{\pi}{\sqrt{\kappa}}$ this triangle is uniquely determined up to congruence. We call the angles $\widetilde{\sphericalangle} x_{i+1}x_ix_{i-1} := \sphericalangle_{x_i}(\widetilde{\gamma}_{i+1}, \widetilde{\gamma}_{i-1})$ (if $\triangle\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3$ exists and is unique; if one of the sides $\overline{\widetilde{x}_i\widetilde{x}_{i-1}}$ or $\overline{\widetilde{x}_i\widetilde{x}_{i+1}}$ has length $\frac{\pi}{\sqrt{\kappa}}$, we set $\widetilde{\sphericalangle} x_{i+1}x_ix_{i-1} = 0$) the *comparison angles*. Observe that if the sides γ_i of $\triangle x_1x_2x_3$ are minimizing geodesics, the comparison triangle and the comparison angles are already defined in terms of the distances between the points x_i.

Assume now that the sectional curvatures of M are bounded below by κ. Hereby we mean $K(\pi) \geq \kappa$ for any 2-plane $\pi \subset T_pM$ and $p \in M$.

Toponogov's Theorem expresses this pointwise bound globally, namely it states that triangles in M are "thicker" than in the corresponding constant curvature case.

Theorem 1.3.1 (Toponogov). *Let* (M, g) *be a complete Riemannian manifold with sectional curvature* $K \geq \kappa$*. Then*

(A) *For any triangle* $\triangle x_1x_2x_3 \subset M$ *with minimizing sides there is a comparison triangle* $\triangle\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3$ *(with* $\ell(\overline{x_ix_{i+1}}) = \ell(\overline{\widetilde{x}_i\widetilde{x}_{i+1}})$*) such that for any point* $w \in \overline{x_2x_3}$ *and the corresponding point* $\widetilde{w} \in \overline{\widetilde{x}_2\widetilde{x}_3}$ *(i.e. the point on* $\overline{\widetilde{x}_2\widetilde{x}_3}$ *for which* $\mathrm{dist}(\widetilde{x}_2, \widetilde{w}) = \mathrm{dist}(x_2, w)$ *and* $\mathrm{dist}(\widetilde{w}, \widetilde{x}_3) = \mathrm{dist}(w, x_3)$ *holds) we have* $\mathrm{dist}(x_1, w) \geq \mathrm{dist}(\widetilde{x}_1, \widetilde{w})$*.*

(B) *For any two minimizing geodesics* $\gamma, \sigma \subset M$ *with* $\gamma(0) = \sigma(0) = x$ *the comparison angle* $\widetilde{\sphericalangle}\gamma(t)x\sigma(s)$ *(in* M^2_κ*) exists and is monotonically decreasing both in* t *and in* s*.*

(C) *For any four distinct points* $x_0, x_1, x_2, x_3 \in M$ *the comparison angles* $\widetilde{\sphericalangle} x_1x_0x_2$*,* $\widetilde{\sphericalangle} x_2x_0x_3$*,* $\widetilde{\sphericalangle} x_3x_0x_1$ *in* M^2_κ *exist and*

$$\widetilde{\sphericalangle} x_1x_0x_2 + \widetilde{\sphericalangle} x_2x_0x_3 + \widetilde{\sphericalangle} x_3x_0x_1 \leq 2\pi.$$

(D) *For any triangle* $\triangle x_1x_2x_3 \subset M$ *with positive side lengths there is a comparison triangle* $\triangle\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3 \subset M^2_\kappa$ *with*

$$\sphericalangle_{x_i}(\overline{x_ix_{i-1}}, \overline{x_ix_{i+1}}) \geq \sphericalangle_{\widetilde{x}_i}(\overline{\widetilde{x}_i\widetilde{x}_{i-1}}, \overline{\widetilde{x}_i\widetilde{x}_{i+1}}) = \widetilde{\sphericalangle} x_{i+1}x_ix_{i-1}.$$

Proof. For a proof of property (D) see [CE] or [Kar] who primarily proves a generalization of property (A) namely that even every *secant* of $\triangle x_1x_2x_3$ is longer than the corresponding secant in the comparison triangle (not just the ones starting at a vertex). However, property (A) as well as (B) can also be deduced from property (D) by the fact that in M^2_κ the length of a side of a triangle with minimizing sides varies monotonically with the opposite angle. Property (C) can be reduced to the Euclidean case by using (D). ∎

One can even refine the assertions of the Theorem: In [CE] it is shown that in (D) instead of claiming that all sides of $\triangle x_1x_2x_3$ are minimizing, it suffices to assume that e.g. the sides $\overline{x_1x_2}$ and $\overline{x_2x_3}$ are minimizing and the length of $\overline{x_3x_1}$ does not exceed $\frac{\pi}{\sqrt{\kappa}}$. The assertion then holds for $i = 1, 3$. Similarly, in (A) it is enough to claim that the sides $\overline{x_1x_2}$ and $\overline{x_1x_3}$ are minimizing and the length of $\overline{x_2x_3}$ is not larger than $\frac{\pi}{\sqrt{\kappa}}$ or the other way round. Analogously, we may generalize (B) to the case where one geodesic is minimizing and the other has length not larger than $\frac{\pi}{\sqrt{\kappa}}$.

It is also possible to formulate a localized version of Toponogov's Theorem: After inspecting the proof, it becomes clear that in order to ensure property (A) we just have to claim that if $\triangle x_1x_2x_3 \subset B_r(x_1)$, the sectional curvatures $K \geq \kappa$ on $B_{3r}(x_1)$ and the ball is relatively compact in M. Analogously, (B) holds if we have the curvature bound on the relatively compact ball $B_r(x) \subset M$ and $\gamma, \sigma \subset B_r(x)$. (C) holds if we have the curvature bound on the compact ball $B_r(x_0)$ and $x_1, x_2, x_3 \in B_r(x_0)$. Last but not least (D) is true if the curvature bound holds on the compact ball $B_r(x_i)$ and $x_{i-1}, x_{i+1} \in B_r(x_i)$.

Toponogov's theorem gives us a tool to synthesize the sectional curvature bound $K \geq \kappa$:

Definition 1.3.2 (Alexandrov space). *A locally compact complete length space (X, d) is called an* Alexandrov space of curvature $\geq \kappa$ *if one of the (equivalent) properties (A), (B) or (C) in Theorem 1.3.1 is satisfied.*
A length space X is said to be locally Alexandrov of curvature $\geq \kappa$ in $x \in X$ *if X is locally compact in x and there is a neighborhood U of x such that property (A), (B) or (C) apply for all triangles, minimizing geodesics resp. points in U.*

In fact, properties (A), (B) and (C) can be shown to be equivalent in locally compact complete length spaces. For a proof see [BGP, §2]. Moreover, with the help of property (B) we can define the *angle* between two geodesics $\gamma, \sigma \subset X$ starting in some point $x \in X$ by

$$\sphericalangle_x(\gamma, \sigma) := \lim_{s,t \to 0} \widetilde{\sphericalangle}\gamma(t)x\sigma(s).$$

It is easy to prove that property (D) holds in Alexandrov spaces.

Using property (C) we easily conclude that the Alexandrov property is closed under Gromov-Hausdorff limits (see chapter 3), i.e. the limit of a sequence of Alexandrov spaces of curvature $\geq \kappa$ is again an Alexandrov space of curvature $\geq \kappa$.

Note that our definition for Alexandrov spaces differs from the definition given in [BGP]. In this paper the authors only assume that the space is *locally* Alexandrov. Next they prove in a globalization theorem that this definition is equivalent to the one stated above.

1.4 Topology and geometry of manifolds with curvature bounded from below

The assumptions $\mathrm{Ric} \geq (n-1)\kappa$ or $K \geq \kappa$ have strong topological and geometric implications. For example recall Toponogov's Theorem 1.3.1 at this point. In the case of bounded Ricci curvature we have the following result (see [dCa, 9 §3]):

Theorem 1.4.1 (Bonnet, Myers). *A complete connected Riemannian manifold (M, g) with $\mathrm{Ric} \geq (n-1)\kappa > 0$ has diameter $\operatorname{diam} M \leq \frac{\pi}{\sqrt{\kappa}}$ and thus is compact.*

Moreover, we conclude that the fundamental group of any compact manifold M with $\mathrm{Ric} > 0$ must be finite.

Theorem 1.4.1 is a direct consequence of a more general result (see [Kar, 1.9.2]):

Theorem 1.4.2 (Bishop, Gromov). *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)\kappa$, $x \in M$ and $x_0 \in M^n_\kappa$. Then the quantity*

$$\frac{\operatorname{vol} B_r^M(x)}{\operatorname{vol} B_r^{M^n_\kappa}(x_0)}$$

is noninreasing in r and its limit for $r \to 0$ is 1.

In the case of nonnegative Ricci curvature, we cannot assume compactness. However, we have the Ricci splitting Theorem (see [CG]):

Theorem 1.4.3 (Cheeger, Gromoll). *Assume that (M^n, g) is a connected Riemannian manifold of nonnegative Ricci curvature that contains a line γ. Then isometrically $M \cong N \times \mathbb{R}$ where N^{n-1} is a complete manifold of nonnegative Ricci curvature. Moreover, the product can be chosen such that γ is parallel to the $\mathbb{R}$ factor (i.e. $\gamma = \{x\} \times \mathbb{R}$ for some $x \in N$).*

The theorem has strong topological implications: Let M be a connected noncompact Riemannian manifold. Let $K \subset M$ be a compact subset. We call the noncompact components of $M \setminus K$ the *ends* of M. Assume now that M carries a complete metric g of nonnegative Ricci curvature. If K can be chosen such as to produce at least two ends $M_1, M_2 \subset M \setminus K$, we can find a sequence of minimizing geodesics γ_k connecting M_1 with M_2 such that the endpoints lie further and further away from K. Since the γ_k have to pass K, they subconverge to a line $\gamma \subset M$ and by the Ricci splitting theorem we must have $M \cong N \times \mathbb{R}$ and the projection of K to N must be N. Hence N is compact. We get that M can only have one or two ends.

In dimension two there are only two compact connected manifolds that admit a metric of positive curvature: the sphere S^2 and the projective space $\mathbb{R}P^2$. If we relax the condition of positive curvature to nonnegative curvature we have to add the torus T^2 and the Klein bottle K^2 to our list. This is a consequence of the Gauß-Bonnet Theorem (see [Ber, Sec 15.7])

Theorem 1.4.4 (Gauß-Bonnet). *Let (M, g) be a closed surface and denote $\chi(M)$ its Euler characteristic. Then*

$$\chi(M) = \frac{1}{4\pi} \int_M S \mathrm{d}\mu.$$

The only noncompact complete 2-manifolds are $\mathbb{R}^2$, $S^1 \times \mathbb{R}$ or the Möbius strip Moe^2 (this is a consequence of the Soul Therem 1.4.8 mentioned below).

In dimension three Richard Hamilton has shown in [Ham1] using the fact that a metric of positive Ricci curvature gets asymptotically round under the Ricci flow:

Theorem 1.4.5 (Hamilton). *If a compact manifold M^3 admits a metric of positive Ricci curvature then it even carries a metric of constant positive sectional curvature. Thus M is diffeomorphic to a spherical space form.*

Recall that a 3 dimensional spherical space form is a manifold that is diffeomorphic to a quotient S^3/Γ of the round 3-sphere where $\Gamma \triangleleft \operatorname{Isom} S^3$ is a discrete subgroup acting freely on S^3. For the classification of spherical space forms see [Wol].

Moreover, by the strong maximum principle (see Theorem 2.5.4) we can show that if M only admits metrics of nonnegative but not of positive Ricci curvature then M has at least one parallel nonzero vector field $X \in \Gamma TM$ with $\mathrm{Ric}(X, X) = 0$. So the 1-form $\alpha := \langle X, \cdot \rangle \in \Omega^1 M$ is harmonic and Hodge theory[1] theory gives us that $H^1(M; \mathbb{R}) \neq 0$.

[1] For an introduction to Hodge theory see [Roe, Chp 6]. However we can also directly prove that $\alpha \neq 0$ in $H^1(M;\mathbb{R})$: Assume $\alpha = \mathrm{d}f$. The parallelity of α implies $\operatorname{Hess} f \equiv 0$, so $f \equiv \text{const}$.

This forces the universal cover $\widetilde{M}$ of M to be noncompact. Since $\pi_1 M$ acts cocompactly on the universal cover, $\widetilde{M}$ contains a line and by the Ricci splitting Theorem 1.4.3 we have $\widetilde{M} \cong N \times \mathbb{R}$ where N^2 has nonnegative curvature. By the strong maximum principle applied to the Ricci flow on N we can assume that N is either positively curved or flat, hence $N \approx S^2$ or $N \cong \mathbb{R}^2$.

We can prove that in the case $N \approx S^2$ the manifold M is diffeomorphic to a metric quotient of the round cylinder[2] $S^2 \times \mathbb{R}$. So in the first case M is diffeomorphic to one of the following manifolds: $S^2 \times S^1$, $S^2 \widetilde{\times} S^1 := S^2 \times S^1 / \sim$ (where $(x, y) \sim (-x, -y)$), $\mathbb{R}P^3 \# \mathbb{R}P^3$ or $\mathbb{R}P^2 \times S^1$.

In the second case, $\widetilde{M}$ is isometric to 3 dimensional Euclidean space. We can apply the Bieberbach Theorem (Corollary 4.1.13 and Proposition 4.2.4 in [Thu]).

Theorem 1.4.6 (Bieberbach). *Let $\Gamma < \operatorname{Isom} \mathbb{R}^n$ be a discrete group of isometries of $\mathbb{R}^n$. Then there is a subgroup $\Gamma' \leq \Gamma$ of finite index and a unique Γ'-invariant affine subspace $V \subset \mathbb{R}^n$ on which Γ' acts cocompactly by translations.*

Thus M is diffeomorphic to a metric quotient of a flat torus T^3 (for a classification of the flat 3-manifolds see [Sco] or [Wol]). Observe that any metric of nonnegative Ricci curvature on one of these manifolds is automatically flat: It is enough to show this statement for the torus T^3. Since $\mathbb{Z}^3 = \pi_1 T^3$ acts cocompactly on the universal cover, $\widetilde{M}$ contains a line and by the Ricci splitting Theorem $\widetilde{M} \cong N \times \mathbb{R}$. Again, N is noncompact and we have a cocompact operation of $\mathbb{Z}^3$ on N. Thus $N \cong \mathbb{R}^2$. For completeness we mention that this fact is an immediate consequence of the Bochner Theorem (see [Bal2, 1.3]):

Theorem 1.4.7 (Bochner). *Let M be a closed and connected Riemannian manifold of nonnegative Ricci curvature. Then $b_1(M) = \dim H^1(M, \mathbb{R}) \leq n$ and equality holds if and only if M is flat.*
Moreover, there are exactly $b_1(M)$ linearly independent parallel vector fields $X_1, \ldots, X_{b_1(M)} \in \Gamma TM$.

The assumption of nonnegative sectional curvature has even stronger implications on the topology of the manifold:

Theorem 1.4.8 (Soul theorem). *Let (M^n, g) be a complete noncompact Riemannian manifold of nonnegative sectional curvature. Then M contains a compact totally geodesic and totally convex submanifold $S \subset M$ whose induced metric has nonnegative Ricci curvature.*
Moreover, M is diffeomorphic to the normal bundle $\nu(S)$ of S in M.
If M has everywhere positive sectional curvature, then the soul S is always a point and M is diffeomorphic to $\mathbb{R}^n$.

We are now able to classify all topological types of noncompact complete Riemannian 3-manifolds M with nonnegative sectional curvature: Let S be a soul of M.

If S is a point, we have $M \approx \mathbb{R}^3$.

If S is one dimensional, it has to be diffeomorphic to S^1. Hence, in this case M is diffeomorphic to a plane bundle over S^1. We conclude that $M \approx S^1 \times \mathbb{R}^2$ or $\mathrm{Moe}^2 \times \mathbb{R}$. In both cases the lift $\widetilde{S}$ of S in the universal cover $\widetilde{M}$ is a line. So by the Ricci splitting Theorem we have $\widetilde{M} \cong \mathbb{R} \times N$, where N is a 2-manifold of nonnegative curvature. Hence $M \cong \mathbb{R} \times N/\Gamma$ where $\Gamma = 1$ or $\Gamma = \langle (s, \Phi) \rangle$ for an $s > 0$ and an isometry $\Phi : N \to N$ that fixes a point.

[2] If N is not isometric to the round S^2 we can either use the result of [Ham2] or the facts that N has no nontrivial orientation preserving isometries without fixed points (Therem 3.7 in [dCa]) and that any orientation preserving isometry of N is isotopic to the identity since it is a Möbius transform on $N \approx \mathbb{C}P^1$ where we choose the induced complex structure on N.

Finally, if S is two dimensional, we have $S \approx S^2, \mathbb{R}P^2, T^2$ or K^2 and M is either diffeomorphic to $S \times \mathbb{R}$ or it is a twisted line bundle over S. In the case $S \approx S^2$ or $\mathbb{R}P^2$ there are the following possibilities: $M \approx S^2 \times \mathbb{R}$, $\mathbb{R}P^2 \times \mathbb{R}$ or $S^2 \widetilde{\times} \mathbb{R} := (S^2 \times \mathbb{R})/\sim$ (where $(a,b) \sim (-a,-b)$). If $S \approx T^2$ or K^2, then the lift $\widetilde{S}$ of S in the universal cover $\widetilde{M}$ is isometric to $\mathbb{R}^2$ and totally geodesic. Hence by the Ricci splitting Theorem M is flat and isometric to a quotient of the Euclidean space $\mathbb{R}^3$ (for a classification of the flat 3-manifolds see again [Wol]).

1.5 Cones

Definition 1.5.1. *Let (N, d_N) be a metric space with* $\operatorname{diam} N \leq \pi$. *Denote $C = \operatorname{Cone}(N) := N \times [0,\infty)/\sim$ where $(\alpha_1, 0) \sim (\alpha_2, 0)$ for all $\alpha_1, \alpha_2 \in N$. We define a metric d_C on C by*

$$d_C((\alpha_1, s_1), (\alpha_2, s_2)) := \sqrt{s_1^2 + s_2^2 - 2 s_1 s_2 \cos d_N(\alpha_1, \alpha_2)}$$

Then (C, d_C) becomes a metric space which we call the (metric) cone over N *and $p := [N \times \{0\}]$ its* tip.

The cone C is called smooth *if $C_0 := C \backslash \{p\}$ carries the structure of a smooth manifold with Riemannian metric g_C whose path metric is locally isometric to d_C.*

Observe that by the above Definition e.g. the set $\{(x_0, x_1, x_2) \in \mathbb{R}^3 \;:\; x_0^2 = x_1^2 + x_2^2, \;\; x_0 \geq 0\} \subset \mathbb{R}^3$ with the restricted metric is not a smooth cone since it is not locally a length space. Furthermore, there are smooth cones C with the property that d_C is not *globaly* isometric to the path metric of g_C (consider for example the case in which $N = S^1$ and $d_N = \frac{1}{2} d_{S^1} + \frac{1}{2}\Phi^* d_{S^1}$ where $\Phi : S^1 \to S^1$ denotes the double cover and d_{S^1} the angle metric, i.e. $d_{S^1}(e^{i\alpha}, e^{i\beta}) = |\arg e^{i(\alpha-\beta)}|$).

In the following we use the scalar function $r : C \to [0,\infty)$ to denote the radial distance from the tip, $r := d_C(p, \cdot) = \mathrm{pr}_2$. For every $\lambda > 0$ there is the canonical homothety $T_\lambda : C \to C$ with $T_\lambda[(\alpha, s)] = [(\alpha, \lambda s)]$. Obviously $T_\lambda^* d_C = \lambda d_C$. Note that $T_{[0,\infty)} x$ is a geodesic ray for all $x \in C_0$. So if C is smooth, $t \mapsto T_t x$ is smooth in t and we may define the vector field ∂_r on C_0 by $(\partial_r)_x := \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}t}|_{t=1} T_t x$. So $\|\partial_r\| = 1$. Moreover, T_{e^t} is the flow of the vector field $Z = r\partial_r$. Observe that we do not a priorily know that ∂_r or Z is smooth.

Proposition 1.5.2. *Let $C = \operatorname{Cone}(N)$ be a smooth cone. Then r and ∂_r are smooth on C_0 and N can be given the structure of a Riemannian manifold with metric g_N such that $C_0 = N \times (0,\infty)$ as a smooth manifold and the Riemannian metric g_C on C_0 has the form*

$$g_C = r^2 g_N + \mathrm{d}r \otimes \mathrm{d}r. \tag{1.1}$$

Moreover, we have

$$\nabla_v(r\partial_r) = v \qquad \text{and} \qquad R(v, \partial_r)\partial_r = 0 \tag{1.2}$$

for any vector $v \in TC_0$.

Proof. Observe that for any $\lambda > 0$ the map $T_\lambda : C \to C$ is smooth on C_0 since it is locally an isometry of the path metrics of (C_0, g_C) and $(C_0, \frac{1}{\lambda^2} g_C)$.

Consider the vector field $Z := r\partial_r$. So for any $x \in C_0$ we have $Z_x = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=1} T_t x$. Then, $\|Z\| = r = \mathrm{pr}_2$. We will show that Z is smooth. This will then imply the smoothness of r and ∂_r.

Let $x \in C_0$. Choose a neighborhood $U \subset C_0$ around x such that between any pair of points $y, z \in U$ there exists a unique minimizing geodesic $\gamma_{y,z} : [0,1] \to C_0$ that depends smoothly on y and z. Consider a small embedding of an $n-1$ ball $\Phi : \mathbb{B}^{n-1} \to U$ with

$0 \mapsto x$ such that the image is transverse to the geodesic $T_{[0,\infty)}x$ in x. Moreover, choose a small $\delta > 0$ such that $T_{[1,1+\delta]}\Phi(\mathbb{B}^{n-1}) \subset U$ and define

$$\Psi : \mathbb{B}^{n-1} \times [0,1] \to C_0 \qquad (q,t) \mapsto \gamma_{\Phi(q),T_{1+\delta}\Phi(q)}(t) = T_{1+t\delta}\Phi(q).$$

Ψ is a local diffeomorphism at $(0,0)$ and carries the (smooth) vector field $(0, \frac{1+t\delta}{\delta}\partial_t)$ on $\mathbb{B}^{n-1} \times [0,1]$ over to the vector field Z. Thus Z is smooth in x.

Since $\partial_r \cdot r = 1$ we conclude that r does not have any critical points on C_0. So if we identify N with $r^{-1}(1)$, we find $C_0 = N \times (0,\infty)$ in the smooth sense. Let $g_N := \iota_N^* g_{C_0}$, where $\iota : N \to C_0$ is the above embedding. It is easy to see that (1.1) holds.

As for assertion (1.2) we observe that for any two vectors $v, w \in T_xC_0$ we have $\langle \nabla_v \partial_r, w\rangle = \langle \nabla_v \nabla r, w\rangle = \nabla^2_{v,w} r$, so $\nabla.\partial_r$ is a symmetric endomorphism and so is $\nabla.Z = \mathrm{d}r \otimes \partial_r + r\nabla.\partial_r$. Hence we get

$$2\langle v,w\rangle = \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} T^*_{e^t} g \right)(v,w) = (\mathcal{L}_Z g)(v,w) = \langle \nabla_v Z, w\rangle + \langle \nabla_w Z, v\rangle = 2\langle \nabla_v Z, w\rangle.$$

This shows the first part of (1.2). As for the second part we extend v to a vector field $V \in \Gamma(TC_0)$ with $V_x = v$ and $[V,Z]_x = 0$ and get

$$R(v,Z)Z = (\nabla_V \nabla_Z Z - \nabla_Z \nabla_V Z)_x = (\nabla_V Z - \nabla_Z V)_x = 0. \qquad \square$$

Chapter 2

Basics on Ricci flow

2.1 Notations and conventions

Let M be an n dimensional manifold and $(g_t)_{t\in I}$ a smooth family of Riemannian metrics parameterized by an interval I. For $t \in I$ denote the Riemannian curvature tensor of (M, g_t) by R_t, the Ricci tensor by Ric_t, the scalar curvature by $S(\cdot, t)$ and the path metric by dist_t. Analogously we define the geometric quantities diam_t, vol_t and the Riemannian measure μ_t. Furthermore, we denote *distance balls* by $B_r(x,t) := B_r^{(M,g_t)}(x) \subset M$. Depending on the situation we sometimes assume $B_r(x,t)$ to lie in $M \times \{t\}$. For $\Delta t \in \mathbb{R}$ we define the *parabolic neighborhood* $P(x,t,r,\Delta t) := B_r(x,t) \times ([t, t+\Delta t] \cap I)$ if $\Delta t \geq 0$ resp. $P(x,t,r,\Delta t) := B_r(x,t) \times ([t+\Delta t, t] \cap I)$ if $\Delta t \leq 0$.

Assume now that the metric g_t satisfies

$$\frac{\partial}{\partial t} g_t = -2\,\mathrm{Ric}_t$$

everywhere on $M \times I$. Then we say that $(M \times I, g_t)$ is a *Ricci flow* or a *Ricci flow solution* that we will also denote by $M \times I$ if there is no chance of confusion. We call I the *time interval* and the elements of I the *times.* For $t \in I$ we denote by $M(t) := (M, g_t)$ or sometimes $(M \times \{t\}, g_t)$ the *time t slice.*

Let (S^2, g) be the round 2-sphere that has constant scalar curvature 1. Unlike in most other texts on differential geometry we will refer to this metric as the *standard round metric.* Note that the metric that is usually refered to as the standard round metric is $\frac{1}{2}g$. It is easy to check that $(S^2 \times (-\infty, 1), (1-t)g)$ is a Ricci flow on a maximal time interval. Observe that the scalar curvature at time t is equal to $\frac{1}{1-t}$. We will refer to this flow as the *standard round shrinking* S^2 and the product flow that is $(S^2 \times \mathbb{R}) \times (-\infty, 0]$ as the *standard round shrinking cylinder.* Analogously we can proceed in higher dimensions: For $n \geq 2$ consider the round metric g on S^n that has scalar curvature 1. Then $(S^3 \times (-\infty, \frac{n}{2}), (1 - \frac{2}{n}t)g)$ is a Ricci flow on a maximal that will be called the *standard round shrinking S^n.*

Let $(\Phi_t)_{t\in I}$ be a smooth family of diffeomorphisms $\Phi_t : M \to M$ and consider smooth families of Riemannian metrics $(g_t)_{t\in I}$ on M that are of the form $g_t = \lambda(t)\Phi_t^* g$ where g is a Riemannian metric on M and $\lambda : I \to \mathbb{R}_+$ a smooth scalar function. If (g_t) satisfies the Ricci flow equation, then we call $(M \times I, g_t)$ a *soliton.* Consider first the case in which Φ_t is the flow of a vector field $V \in \Gamma TM$ and $\lambda \equiv 1$. Then the Ricci flow equation for (g_t) is equivalent to

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g = 0.$$

If V is the gradient of a function f with respect to the metric g, we call $M \times I$ or (M, g) a *gradient soliton* or a *steady gradient soliton.* Observe that in this case we have

$$\mathrm{Ric} + \nabla^2 f = 0.$$

On the other hand, it can be shown that if Φ_t is the flow of the time dependend vector field $\frac{1}{t}V$ (for some vector field V on M) and if $\lambda(t) = ct$ for some $c \neq 0$, then (g_t) is a Ricci flow if and only if

$$\operatorname{Ric} + \frac{c}{2}\mathcal{L}_V g + \frac{c}{2} g = 0.$$

In the case in which $V = \nabla f$ this becomes

$$\operatorname{Ric} + c\nabla^2 f + \frac{c}{2} g = 0.$$

If $c > 0$, then $I \subset (0, \infty)$ and we call $M \times I$ or (M, g) an *expanding gradient soliton.* On the other hand, if $c < 0$, we have $I \subset (-\infty, 0)$ and $M \times I$ or (M, g) is called a *gradient shrinking soliton.*

2.2 Parabolic rescaling

Obviously, if $(M \times I, g_t)$ is a Ricci flow, then so is $(M' \times I', g'_{t'}) = (M \times (\lambda^2 I), \lambda^2 g_{\lambda^{-2}t})$ for $\lambda > 0$ (here $\lambda^2 I$ denotes the scaled interval, e.g. $\lambda^2 I = [\lambda^2 T_1, \lambda^2 T_2]$ if $I = [T_1, T_2]$). We call this process *parabolic rescaling* and λ the *scaling factor.* Moreover, we write $M' \times I' = \lambda(M \times I)$.

It is important to keep in mind how geometric quantities rescale: The scalar curvature $S'(x, t')$ of the rescaled Ricci flow equals $\lambda^{-2} S(x, \lambda^{-2}t')$. Analogously, the sectional curvature $K'_{x,t'}$ is equal to $\lambda^{-2} K_{x,\lambda^{-2}t'}$. The Ricci tensor viewed as element of $\Gamma(\operatorname{End}^{\mathrm{sym}} \Lambda_2 TM)$ and the Riemmanian curvature tensor $R \in \Gamma(T_1^3 TM)$ do not change under parabolic rescaling except for the dilation in time. Finally, for the Riemannian measure we have $\mu'_{t'} = \lambda^n \mu_{\lambda^{-2}t'}$.

2.3 Distance distortion estimates

We mention two results that give bounds on the distortion of the distance function in terms of bounds on the curvature. For the proofs see [KL, Sec 27].

Theorem 2.3.1. *Let $M \times [t_1, t_2]$ be a Ricci flow on an n dimensional manifold. Suppose that the Ricci curvature satisfies $(n-1)\kappa_{\min} \leq \operatorname{Ric} \leq (n-1)\kappa_{\max}$. Then*

$$e^{-(n-1)\kappa_{\max}(t_2 - t_1)} \leq \frac{\operatorname{dist}_{t_2}(x, y)}{\operatorname{dist}_{t_1}(x, y)} \leq e^{(n-1)\kappa_{\min}(t_2 - t_1)}$$

for any distinct $x, y \in M$.

This theorem implies the following fact: If $M \times I$ is a Ricci flow with bounded Ricci curvature and one time slice $M(t)$ is complete, then all time slices are complete.

Theorem 2.3.2. *There is a constant $C_n < \infty$ depending only on the dimension n such that the following holds:*
Let $M \times I$ be a Ricci flow on an n dimensional manifold and let $t_0 \in I$ be not the initial time. Assume that $M(t_0)$ is complete and satisfies $\operatorname{Ric} \leq (n-1)\kappa$. Then in the barrier sense we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t^-}\right|_{t=t_0} \operatorname{dist}_t(x, y) \geq -C_n \sqrt{\kappa}$$

for any $x, y \in M$.

Observe that the bound is independent of the distance itself.

2.4 Evolution of geometric quantities

It is easy to deduce that the Riemannian measure satisfies

$$\dot{\mu}_t = -S\mu_t$$

under the Ricci flow.

We will now briefly discuss the results obtained in [LB1]. Consider a Ricci flow $M \times I$ on an n dimensional manifold M. Let $\pi : M \times I \to M$ be the natural projection and consider the pullback $T^{\text{spat}}(M \times I) := \pi^* TM$ of the tangent bundle TM and pullbacks $T^k_l T^{\text{spat}}(M \times I) = T^k_l \pi^* TM = \pi^* T^k_l TM$ of higher tensor bundles. There is a natural *metric* connection $\bar{\nabla}$ on $T^{\text{spat}}(M \times I)$ with respect to the pulled back Riemannian metric: Let $T := (0, \partial_t) \in \Gamma T(M \times I)$. For spatial vector fields $X, Y \in \Gamma T^{\text{spat}}(M \times I) \subset \Gamma T(M \times I)$ we define $\bar{\nabla}$ by

$$\bar{\nabla}_X Y := \nabla_X Y \qquad \text{and} \qquad \bar{\nabla}_T X = \frac{\partial}{\partial t} X - \operatorname{Ric}(X).$$

It is easy to check that $\bar{\nabla} g = 0$. If $\mathrm{d}t \in \Omega^1(M \times I)$ denotes the pullback of the 1-form $\mathrm{d}t \in \Omega^1(I)$ under the projection $M \times I \to I$, we can write

$$\bar{\nabla} = \nabla' - \operatorname{Ric} \otimes \mathrm{d}t$$

where ∇' denotes the connection on $\pi^* TM$ which is ∇ on the M factor of $M \times I$ and $\frac{\partial}{\partial t}$ on the second. The connection $\bar{\nabla}$ naturally induces connections on the higher tensor bundles $T^k_l T^{\text{spat}}(M \times I)$ wich we will also denote by $\bar{\nabla}$.

Now it is very convenient to express the $\bar{\nabla}_T$ derivative of the Riemannian curvature (as derived in [LB1]): For two vector fields $X, Y \in \Gamma T^{\text{spat}}(M \times I)$ we have

$$(\bar{\nabla}_T R)(X, Y) = (\triangle R)(X, Y) + \operatorname{tr} R(R(X, Y)\cdot, \cdot) - 2 \operatorname{tr} [R(X, \cdot), R(Y, \cdot)] . \tag{2.1}$$

Note that one major advantage of considering the $\bar{\nabla}_T$ derivative instead of ∇'_T is that the evolution of R takes this form no matter whether we consider the R to be a $(1,3)$ tensor, a $(2,2)$ tensor etc. We can even express (2.1) in a more elegant way: Let $\hat{R} \in \Gamma \operatorname{End}^{\text{sym}}(\Lambda^2 T^{\text{spat}} M)$ be the Riemannian curvature operator. Then we have

$$\bar{\nabla}_T \hat{R} = \triangle \hat{R} + 2\hat{R}^2 + 2\hat{R}^{\#}.$$

Here $\hat{R}^{\#} = \hat{R} \# \hat{R} = \operatorname{ad} \circ (R \wedge R) \circ \operatorname{ad}^*$ where $\operatorname{ad} : \Lambda^2(\Lambda^2 T^{\text{spat}} M) \cong \Lambda^2 so(n) \to so(n)$ denotes the Lie bracket (see [BW] for more details).

By contracting we can deduce the evolution equation for the Ricci curvature

$$\bar{\nabla}_T \operatorname{Ric} = \triangle \operatorname{Ric} + 2 \sum_i R(\cdot, e_i)(\operatorname{Ric} e_i)$$

for any orthonormal frame (e_i). The evolution of the scalar curvature becomes

$$\dot{S} = \frac{\partial}{\partial t} S = \triangle S + 2 \| \operatorname{Ric} \|^2. \tag{2.2}$$

As derived in [LB1], the evolution of the norms of higher curvature derivatives takes the following form:

$$\left(\|\nabla^l R\|^2 \right)^{\cdot} = \triangle \left(\|\nabla^l R\|^2 \right) - 2\|\nabla^{l+1} R\|^2 + \sum_{\substack{i+j=l,\\ i,j \geq 0}} \nabla^i R * \nabla^j R * \nabla^k R \tag{2.3}$$

for $l \geq 0$. Here $\nabla^i R * \nabla^j R * \nabla^k R$ denotes some unspecified contraction of the tensor $\nabla^i R \otimes \nabla^j R \otimes \nabla^k R$ that is equivariant under the action of the orthogonal group.

2.5 Maximum principles

In order to estimate solutions to equations of heat type we can use the *weak maximum principle.* We first mention an easy result (for a proof see e.g. [CLN, Sec 2.3]):

Theorem 2.5.1 (Weak maximum principle, baby case). *Let (M, g) be a compact Riemannian manifold and $u : M \times [0, T_0) \to \mathbb{R}$ be a smooth scalar function satisfying*

$$\dot{u} \geq \triangle u.$$

Now if $u(\cdot, 0) \geq 0$ on M, then $u \geq 0$ everywhere on $M \times [0, T_0)$.

The assertion stays true if we just require the differential inequality to hold in the barrier sense. We can refine this result to the case in which the solution u lives in a vector bundle (see [Ham3, Lem 8.1] or [Lee2])

Theorem 2.5.2 (Weak maximum principle, bundle case). *Let M be a compact manifold with boundary and $(g_t)_{t \in [0,T_0)}$ a smooth family of Riemannian metrics. Furthermore, let $E \to M \times [0, T_0)$ be a Euclidean vector bundle equipped with a metric connection ∇. Consider a closed subbundle $C \subset E$ of convex sets $C_{x,t} \subset E_{x,t}$ that is parallel in spatial direction and a smooth vertical vector field $\Phi \in \Gamma TE$ that has the property that the flow of the vector field $\Phi + \nabla_T$ preserves C (here ∇_T denotes the vector field on E that is horizontal with respect to ∇_T and projects down to the vector field T on M).*
Now if $u \in \Gamma(M \times [0, T_0), E)$ satisfies the PDE

$$\nabla_T u = \triangle u + \Phi(u, t)$$

and if $u \in C$ on $\partial_{\text{par}}(M \times [0, T_0)) := M \times \{0\} \cup \partial C \times [0, T_0)$, then $u \in C$ everywhere on $M \times [0, T_0)$.

There are ways of getting rid of the assumption that M be compact. However, in this case we have to assume that the solution u satisfies some bounds depending on the distance to a basepoint (see [CLN, Sec 7.4] or [Bam1]).

We now discuss the equality case:

Theorem 2.5.3 (Strong maximum principle, baby case). *Let (M, g) be a not necessarily compact Riemannian manifold and $u : M \times [0, T) \to \mathbb{R}$ a smooth scalar function satisfying*

$$\dot{u} \geq \triangle u \qquad \text{and} \qquad u \geq 0.$$

If $u(x, t) = 0$ for some $x \in M$ and $t > 0$, then we already have $u = 0$ on $M \times [0, t]$.

In the bundle case we have (see again [Lee2])

Theorem 2.5.4 (Strong maximum principle, bundle case). *Let M be a not necessarily compact manifold and $(g_t)_{t \in [0,T_0)}$ a smooth family of Riemannian metrics. Furthermore, let $E \to M \times [0, T_0)$ be a Euclidean vector bundle equipped with a metric connection. Consider a closed subbundle $C \subset E$ of convex sets $C_{x,t} \subset E_{x,t}$ that is parallel in space* and *(!) time direction and a vertical vector field $\Phi \in \Gamma TE$ as in Theorem 2.5.2. Assume that $u \in \Gamma(M \times [0, T_0), E)$ satisfies*

$$\nabla_T u = \triangle u + \Phi(u, t) \qquad \text{and} \qquad u \in C \qquad \text{everywhere on } M \times [0, T_0)$$

Now if $u(x, t) \in \partial C$ for some $x \in M$ and $t > 0$, then we have $u \in \partial C$ everywhere on $M \times [0, t]$.
Moreover, for any $(x', t') \in M \times [0, t)$ the vector $\Phi(u(x', t'))$ and the Hessian $\operatorname{Hess} u$ *are perpendicular to any normal vector N of $C_{x',t'}$. (Here, a vector $N \in E_{x',t'}$ is said to be normal to $C_{x',t'}$ at $u(x', t')$ if $\langle v, N \rangle \leq \langle u(x', t'), N \rangle$ for all $v \in C_{x',t'}$.)*

We mention some useful applications of the weak and strong maximum principle:

Corollary 2.5.5. *Let $M \times [0,T)$ be a Ricci flow on a compact n-dimensional manifold M and suppose that $S_{\min} \leq S(\cdot,0) \leq S_{\max}$ on M. Then*

$$\frac{1}{\frac{1}{S_{\min}} - \frac{2}{n}t} \leq S(\cdot,t)$$

and if $\mathrm{Ric} \geq 0$ *everywhere,*

$$S(\cdot,t) \leq \frac{1}{\frac{1}{S_{\max}} - 2t}.$$

Corollary 2.5.6. *Consider a Ricci flow $M \times [0,T)$ on a compact manifold M. If the curvature operator at time* 0 *is everywhere nonnegative definite, then this property is preserved under the Ricci flow.*
In particular, in dimension 3 *nonnegative sectional curvature is preserved.*

Corollary 2.5.7. *Let $M \times [0,T]$ be a Ricci flow on an n dimensional manifold. Assume that the curvature operator $\hat{R}$ is everywhere nonnegative definite and $M(T) \cong N \times \mathbb{R}$, where (N, g_N) is an $n-1$ dimensional Riemannian manifold. Then the splitting already existed before T and the Ricci flow $M \times [0,T]$ is of product form $(N \times \mathbb{R}) \times [0,T]$ where $N \times [0,T]$ denotes a Ricci flow on N.*

Corollary 2.5.8. *Consider a Ricci flow $M \times [0,T)$ on a compact* 3*-manifold M. If* $\mathrm{Ric} \geq 0$ *and* $\mathrm{Ric} \geq cS$ *for some $c \geq 0$ at time* 0 *then this condition is preserved under the Ricci flow.*
Moreover, if $c > 0$ and for some $t > 0$ there is no $c' > c$ such that $\mathrm{Ric} \geq c'S$ *at time t, then $M \times [0,T)$ is the Ricci flow of a metric quotient of the round S^3.*

Proof. The first statement can be found in [Ham1, Thm 9.6]. For the second statement we apply the strong maximum principle. Analyzing the calculations in the reference, we find that either $\mathrm{Ric} = cS\langle\cdot,\cdot\rangle$ everywhere on $M \times [0,t]$ (this implies that $M(0)$ has constant nonnegative curvature) or Ric has a zero eigenvalue in which case $M \times [0,T)$ must be flat. But flat metrics do not satisfy the condition of the last statement. □

2.6 Shi's estimates

We can derive a smoothing property of the Ricci flow from the weak maximum principle and (2.3):

Theorem 2.6.1 (global Shi estimates). *Consider a Ricci flow $M \times [0,T)$ on a compact manifold M. Assume that $\|R\| \leq D$ on $M \times [0,T)$. Then we have*

$$\|\nabla^l R\| \leq C_l \frac{D}{t^{l/2}} \qquad \text{on} \qquad M \times \left([0, \tfrac{1}{D}] \cap [0,T)\right)$$

for any $l \geq 0$ where the C_l are constants that only depend on l and the dimension n.

Observe that we don't assume bounds on higher curvature derivatives at time 0.

Proof. For a proof see [Ham5, Sec 7] or [LB3]. □

A localization of this statement reads as follows:

Theorem 2.6.2 (local Shi estimates). *Let $D < \infty$, $T \le \frac{1}{D}$ and $r \ge \frac{1}{\sqrt{D}}$. Consider a Ricci flow $M \times [0, T)$ on a not necessarily complete manifold M and a point $p \in M$. Assume that $B_r(p, 0) \subset M$ is relatively compact and $\|R\| < D$ on $B_r(p, 0) \times [0, T]$. Then for all $l \ge 0$*

$$\|\nabla^l R\|(p, t) \le C_l \frac{D}{t^{l/2}} \qquad \text{for all} \qquad t \in (0, T].$$

Here C_l again depends only on l and the dimension n.

Proof. A proof of this statement can be found in [Shi]. For a more readable exposition see [Ham5, Sec 13] or [LB4]. □

2.7 Short and long time existence

Theorem 2.7.1 (short time existence and uniqueness). *Let (M, g_0) be a compact Riemannian manifold. Then there is a $\delta > 0$ and a Ricci flow $(M \times [0, \delta), g_t)$ with initial metric g_0.*
Moreover, the solution is unique in the following sense: If $(M \times [0, \delta'), g'_t)$ is another solution with initial metric g_0, then g_t and g'_t coincide on $[0, \delta) \cap [0, \delta')$.

A proof of this result can be found in [Ham1]. However, this proof makes use of deep analytical tools since the Ricci flow equation is not strictly parabolic. By a method known as "DeTurck's trick" it is possible to reduce the problem to a strictly parabolic one (see [DeT]). Now the existence and uniqueness follows from standard PDE techniques. For an exposition of the methods used to prove Theorem 2.7.1 see [Hei2] or [Bam2].

Using the global Shi estimates it is easy to discuss the long time existence statement:

Theorem 2.7.2 (long time existence). *Let $M \times [0, T)$ be a Ricci flow on a compact manifold defined on a maximal time interval $[0, T)$. Then $\max_M \|R_t\| \to \infty$ for $t \nearrow T$.*

Proof. Applying the weak maximum principle to $\|R\|^2$ (see (2.3) for the case $l = 0$), it is easy to see that either $\max_M \|R_t\|$ stays bounded in t or $\max_M \|R_t\| \to \infty$ for $t \nearrow T$. Assume that $\max_M \|R_t\|$ is bounded. By the global Shi estimates (Theorem 2.6.1) we find that the curvature derivatives $\|\nabla^l R\|$ are each bounded on $M \times [0, T)$. Fix an arbitrary Riemannian metric $\tilde{g}$ on M and denote by $|\cdot|$ its associated norm as well as by $\tilde{\nabla}$ its Levi-Civita connection. By Lemma 3.3.2 in chapter 3 we conclude that the derivatives $|\tilde{\nabla}^l g_t|$ are uniformly bounded on $M \times [0, T)$. So the family of metrics g_t smoothly converges to a metric g_T on M. This shows that we can extend the Ricci flow on $M \times [0, T)$ to a larger time interval $[0, T]$ and by Theorem 2.7.1 to an even larger time interval $[0, T + \delta)$. A contradiction. □

2.8 A Harnack inequality for the Ricci flow

We can derive a differential inequality similar to the Harnack inequality from [LY].

Theorem 2.8.1. *Let $M \times (0, T)$ be a Ricci flow with complete time slices and nonnegative curvature operator. Assume furthermore that the curvature is bounded on compact time intervals. Then for any $(x, t) \in M \times (0, T)$ and $v \in T_x M$*

$$\frac{\partial S}{\partial t}(x, t) + \frac{S}{t} + 2\langle \nabla S, v \rangle + 2 \operatorname{Ric}(v, v) \ge 0.$$

Integrating this inequality appropriately yields:

Theorem 2.8.2. *Under the same assumptions of the Theorem above let* $(x_1, t_1), (x_2, t_2) \in M \times (0, T)$ *such that* $t_1 < t_2$. *Then*

$$S(x_2, t_2) \geq \frac{t_1}{t_2} \exp\left(-\frac{\mathrm{dist}^2_{t_1}(x_1, x_2)}{2(t_2 - t_1)}\right) S(x_1, t_1).$$

For a proofs of both Theorems see [Ham7] or [Mül].

2.9 Hamilton-Ivey pinching

We have seen in Corollary 2.5.6 that in dimension 3 nonnegative sectional curvature is preserved under the Ricci flow. Other lower bounds on the sectional curvature are not preserved. But we have

Theorem 2.9.1 (Hamilton-Ivey pinching). *Let* $M \times [T_1, T_2)$ *be a Ricci flow on a compact manifold* M *and* $\varphi^{-1} > -T_1$. *Consider the following time dependent property: For* $x \in M$ *and* $t \in [0, T)$ *there is an* $X > 0$ *with* $K(x, t) \geq -X$ *such that*

$$S(x, t) \geq X(\log X + \log(\varphi^{-1} + t) - 3) \qquad \textit{and} \qquad S(x, t) \geq -\frac{3}{\varphi^{-1} + t}.$$

Now if all points on M *satisfy the property at time* T_1, *then so do the points at all times of* $[T_1, T_2)$.

Proof. The result is a parabolically rescaled version of [Ham6, Thm 4.1]. □

Observe that the Theorem asserts that although there is no lower bound on the sectional curvature, the ratio of the lowest sectional curvature and the scalar curvature S goes to zero for $S \to \infty$. So if we have an upper bound on the scalar curvature, then we have a lower bound on the sectional curvature. This in turn implies that we have an upper bound on the sectional curvature.

2.10 Ricci flows on cones

Lemma 2.10.1. *Let* (C, p) *be a smooth cone and* $U \subset C_0$ *open. Suppose the metric* g_C *on* U *is the final time slice of a Ricci* g_t *flow on* $U \times (-\varepsilon, 0]$ *such that* g_t *has nonnegative curvature operator for all* $t \in [-\varepsilon, 0]$. *Then* C *is locally flat in* U.

Proof. By Proposition 1.5.2 we know that $\mathrm{Ric}(\partial_r, \partial_r) = 0$ on U. Now, we apply the strong maximum principle to the evolution of the curvature operator (compare also with the proof of Corollary 2.5.7) and get that for any point $x \in U$ there is a neighborhood $V \subset U$ such that for any $z \in V$ there is a parallel vector field $P(z) \in \Gamma(TV)$ with $P_z(z) = (\partial_r)_z$ and $\mathrm{Ric}(P(z), P(z)) = 0$ on V. Let $N_x \subset T_xU$ be the null space of Ric. Since $\mathrm{Ric} \geq 0$ we have $P_x(z) \in N_x$ for any $z \in V$. So for any $v \in T_xU$ we have again by Proposition 1.5.2

$$N_x \ni (\mathrm{d}P_x)_x(v) = (\mathrm{d}P_x)_x(v) + (\nabla_v P)_x(x) = (\mathrm{d}_y(P_y(y)))_x(v) = (\nabla_v \partial_r)_x = \frac{1}{r}v.$$

We conclude $N_x = T_xU$. Hence $K \equiv 0$ on U. □

Chapter 3

Geometric compactness theorems

In the following, unless otherwise stated, metric spaces are always assumed to be complete and proper (i.e. bounded sets are totally bounded). Note that by Proposition 1.2.4 this property is always fulfilled for complete, locally compact length spaces.

Further representations concerning the first two sections can be found in [Gro] and [Ham4]. However, the reader is advised to familiarize with the language developed in these sections.

3.1 Gromov-Hausdorff convergence & compactness

3.1.1 The compact case

Consider a sequence (X_k) of compact metric spaces. We say that (X_k) converges to a compact metric space X_∞ in the Gromov-Hausdorff sense if one of the following (equivalent) conditions is satisfied:

(A) For any $\varepsilon > 0$ there are an $N = N(\varepsilon)$ and ε-nets $\{x_\infty^{(1)}, \ldots, x_\infty^{(m)}\} \subset X_\infty, \{x_k^{(1)}, \ldots, x_k^{(m)}\} \subset X_k$ for $k > N$ such that

$$\left| d\left(x_k^{(i)}, x_k^{(j)}\right) - d\left(x_\infty^{(i)}, x_\infty^{(j)}\right) \right| < \varepsilon \qquad \text{for all } 1 \le i, j \le m.$$

(B) There are a sequence $\varepsilon_k \to 0$ and ε_k-isometries $\Phi_k : X_\infty \to X_k$ with $B_{\varepsilon_k}(\operatorname{Im}\Phi_k) = X_k$ (here $B_\cdot(\cdot)$ denotes the tubular neighborhood), where we call a map $\Phi : A \to B$ between two metric space A and B an *ε-isometry* if

$$\|\Phi^* d_B - d_A\|_\infty < \varepsilon.$$

(C) We have $d_{\mathrm{GH}}(X_k, X_\infty) \to 0$, where d_{GH} denotes the *Gromov-Hausdorff distance*, i.e. for two arbitrary (not necessarily complete or proper) metric spaces A and B

$$d_{\mathrm{GH}}(A,B) := \inf\left\{ d_{\mathrm{H}}^Z(\iota_1 A, \iota_2 B) \ : \ \begin{array}{l} Z \text{ arbitrary metric space, } \iota_1 : A \to Z, \\ \iota_2 : B \to Z \text{ isometric embeddings} \end{array} \right\} \in [0,\infty]$$

Here $d_{\mathrm{H}}^Z(U,V)$ denotes the *Hausdorff distance* in Z, i.e.

$$d_{\mathrm{H}}^Z(U,V) := \inf\left\{ r \ : \ B_r(U) \supset V \text{ and } B_r(V) \supset U \right\}.$$

(D) There are a metric space Z (not necessarily complete or proper) and isometric embeddings $\iota_k : X_k \to Z$, $\iota_\infty : X_\infty \to Z$ with respect to which $d_{\mathrm{H}}(X_k, X_\infty) \to 0$.

Condition (A) is very convenient if one wishes to construct the Gromov-Hausdorff limit of a sequence of metric spaces. Condition (C) is useful to formalize the concept of Gromov-Hausdorff convergence: it enables us to endow the space $\mathcal{X}_{\mathrm{cp}}$ of isometry classes of compact metric spaces[1] with a metric under which convergence is equivalent to Gromov-Hausdorff convergence. It also immediately gives us that the Gromov-Hausdorff limit of a sequence of compact metric spaces is unique. Conditions (B) and (D) are often used for proving that certain geometric properties of the spaces in the sequence carry over to the limit. Moreover, observe that (B) and (D) give a tool to fix a particular "way of convergence" that may be worked with lateron. So for example after fixing the sequence (Φ_k) of ε_k-isometries or the embeddings (ι_k) and ι_∞ we can define what it means to say that a sequence of points $y_k \in X_k$ or a sequence of maps $\varphi_k : X_k \to A, A \to X_k$ or $X_k \to X_k$ (for a metric space A) converges to a point $y_\infty \in X_\infty$ resp. a map $\varphi_\infty : X_\infty \to A, A \to X_\infty$ or $X_\infty \to X_\infty$. For example, if $\gamma_k : I \to X_k$ for $I \subset \mathbb{R}$ an interval are minimizing geodesics converging to some map $\gamma_\infty : I \to X_\infty$, then γ_∞ is itself a minimizing geodesic.

For an arbitrary metric space A denote by $N_\varepsilon^A \in \mathbb{N}_0 \cup \{\infty\}$ the minimal cardinality of closed ε-nets that is to say subsets $T \subset A$ such that any point $a \in A$ has distance $\leq \varepsilon$ to a point $t \in T$ (we could also define N_ε^A to be the minimal cardinality of ε-nets, but this is not a closed condition). Note that $N_\varepsilon^A < \infty$ if A is totally bounded.

Choose a function $\widetilde{N}_\cdot : \mathbb{R}_+ \to \mathbb{N}$, a constant $\widetilde{D} < \infty$ and set

$$\mathcal{X}_{\widetilde{N},\widetilde{D}} := \{X \in \mathcal{X}_{\mathrm{cp}} \ : \ N_\varepsilon^X \leq \widetilde{N}_\varepsilon \quad \text{for all } \varepsilon > 0 \text{ and } \quad \operatorname{diam} X \leq \widetilde{D}\}$$

The following fact is known as Gromov-Hausdorff compactness

Theorem 3.1.1 (Gromov-Hausdorff compactness for compact metric spaces). *The space $(\mathcal{X}_{\widetilde{N},\widetilde{D}}, d_{\mathrm{GH}})$ is compact. In other words, any sequence (X_k) of compact metric spaces satisfying*
(i) $N_\varepsilon^{X_k} \leq \widetilde{N}_\varepsilon$ *for all* $\varepsilon > 0$ *and*
(ii) $\operatorname{diam} X_k \leq \widetilde{D}$
for all k has a subsequence (X_{k_j}) that converges to a compact metric space X_∞ with the same properties (i) and (ii).

If a sequence of metric spaces (X_k) satisfies property (i) then we also say that X_k is *uniformly totally bounded.* Note that by the following Proposition for any sequence of compact n dimensional Riemannian manifolds (M_k, g_k) with $\mathrm{Ric}_k \geq (n-1)\kappa$ and $\operatorname{diam} M_k \leq \widetilde{D}$ (for any $n \in \mathbb{N}$, $\kappa \in \mathbb{R}$ and $\widetilde{D} < \infty$) there is a subsequence (M_{k_i}) that converges to some compact metric space. Here we consider the M_{k_i} as metric spaces equipped with the path metric.

Proposition 3.1.2. *For any $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$ there is a function $\widetilde{N}_\cdot : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{N}$ such that if (M,g) is an n dimensional Riemannian manifold with* $\mathrm{Ric} \geq (n-1)\kappa$, *$x \in M$ and $B_{2r+\varepsilon/2}(x)$ is relatively compact, we have $N_\varepsilon^{B_r(x)} \leq \widetilde{N}_\varepsilon(r)$ for all $\varepsilon, r > 0$.*

Proof. Fix $\varepsilon, r > 0$. Choose $k \in \mathbb{N}$ maximal with the property that there are points $y_1, \ldots, y_k \in B_r(x)$ such that the balls $B_{\varepsilon/2}(y_1), \ldots, B_{\varepsilon/2}(y_k)$ are pairwise disjoint. It is easy to see that $\{y_1, \ldots, y_k\}$ is an ε-net for $B_r(x)$ and $N_\varepsilon^{B_r(x)} \leq k$. We want to estimate k from above. By the Bishop-Gromov Theorem 1.4.2 we have for all $1 \leq i \leq k$

$$\frac{\operatorname{vol} B_{\varepsilon/2}(y_i)}{\operatorname{vol} B^{M_\kappa^n}_{\varepsilon/2}(x_0)} \geq \frac{\operatorname{vol} B_{2r+\varepsilon/2}(y_i)}{\operatorname{vol} B^{M_\kappa^n}_{2r+\varepsilon/2}(x_0)} \geq \frac{\operatorname{vol} B_{r+\varepsilon/2}(x)}{\operatorname{vol} B^{M_\kappa^n}_{2r+\varepsilon/2}(x_0)}$$

[1] Note that we may in fact speak of the set $\mathcal{X}_{\mathrm{cp}}$ since every metric space satisfying the above conditions has cardinality $\leq \aleph_1$. However, this is not important in the following theory.

where M^n_κ denotes the n dimensional model space of constant curvature κ and $x_0 \in M^n_\kappa$ a basepoint. So

$$\frac{\operatorname{vol} B_{\varepsilon/2}(y_i)}{\operatorname{vol} B_{r+\varepsilon/2}(x)} \geq \frac{\operatorname{vol} B^{M^n_\kappa}_{\varepsilon/2}(x_0)}{\operatorname{vol} B^{M^n_\kappa}_{2r+\varepsilon/2}(x_0)}.$$

Summing this inequality over i yields

$$1 \geq \frac{\sum_{i=1}^k \operatorname{vol} B_{\varepsilon/2}(y_i)}{\operatorname{vol} B_{r+\varepsilon/2}(x)} \geq k \frac{\operatorname{vol} B^{M^n_\kappa}_{\varepsilon/2}(x_0)}{\operatorname{vol} B^{M^n_\kappa}_{2r+\varepsilon/2}(x_0)}$$

hence $k \leq \frac{\operatorname{vol} B^{M^n_\kappa}_{2r+\varepsilon/2}(x_0)}{B^{M^n_\kappa}_{\varepsilon/2}(x_0)}$ and the Proposition follows. □

3.1.2 The noncompact case

We now relax the condition that the X_k are compact but still assume completeness and properness. In order to assure uniqueness of the limit we have to choose basepoints $x_k \in X_k$. Now, we say that the sequence (X_k, x_k) of pointed metric spaces converges to a (complete and proper) pointed metric space (X_∞, x_∞) if one of the following equivalent conditions is satisfied:

(A′) For any r and any $\varepsilon > 0$ there are $N = N(r, \varepsilon)$ and ε-nets $\{x_k = x_k^{(0)}, x_k^{(1)}, \ldots, x_k^{(m)}\} \subset B_r^{X_k}(x_k), \{x_\infty = x_\infty^{(0)}, x_\infty^{(1)}, \ldots, x_\infty^{(m)}\} \subset B_r^{X_\infty}$ such that

$$\left| d\left(x_k^{(i)}, x_k^{(j)}\right) - d\left(x_\infty^{(i)}, x_\infty^{(j)}\right)\right| < \varepsilon \qquad \text{for all } 0 \leq i, j \leq m.$$

(B′) There are sequences $r_k \to \infty, \varepsilon_k \to 0$ and ε_k-isometries $\Phi_k : B^{X_\infty}_{r_k}(x_\infty) \to B^{X_k}_{r_k}(x_k)$ such that $B_{\varepsilon_k}(\operatorname{Im}\Phi_k) \supset B^{X_k}_{r_k}(x_k)$ and $d_k(\Phi_k(x_\infty), x_k) < \varepsilon_k$.

(D′) There is a metric space Z (not necessarily complete or proper) and isometric embeddings $\iota_k : X_k \to Z$, $\iota_\infty : X_\infty \to Z$ such that $\iota_k(x_k) \to \iota_\infty(x_\infty)$ and $d_{\mathrm{H}}(X_k \cap U, X_\infty \cap U) \to 0$ for any bounded open set $U \subset Z$.

It is easy to see that the limit (X_∞, x_∞) of a sequence of pointed metric spaces (X_k, x_k) is unique if we require completeness.

Let (A, a) be an arbitrary pointed metric space. For $\varepsilon, r > 0$ we write $N^{(A,a)}_\varepsilon(r) := N_\varepsilon^{B_r^A(a)}$. Choose a function $\widetilde{N}.(\cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{N}$. Now the Gromov-Hausdorff compactness statement reads as follows:

Theorem 3.1.3 (Gromov-Hausdorff compactness for pointed metric spaces). *Any sequence (X_k, x_k) of complete and proper pointed metric spaces satisfying $N^{(X_k,x_k)}_\varepsilon(r) \leq \widetilde{N}_\varepsilon(r)$ for all $\varepsilon, r > 0$ and $k \in \mathbb{N}$ has a subsequence that converges to some complete and proper pointed metric space (X_∞, x_∞).*

Again by Proposition 3.1.2 we conclude that any sequence (M_k, x_k) of complete pointed Riemannian manifolds with $\operatorname{Ric}_k \geq (n-1)\kappa$ (for some $\kappa \in \mathbb{R}$) subconverges to a complete and proper metric space (X_∞, x_∞). We can even be more general: If there is a function $\kappa : [0, \infty) \to \mathbb{R}$ such that $\operatorname{Ric}_k(x) \geq (n-1)\kappa(\operatorname{dist}(x, x_k))$ for all k and $x \in M_k$, then (M_k, x_k) subconverges to a pointed metric space (X_∞, x_∞).

For completeness we mention the following easy fact: If (X_k, x_k) Gromov-Hausdorff converges to some metric space (X_∞, x_∞) and if $\gamma_k : [a_k, b_k] \to X_k$ ($a_k \leq 0 \leq b_k$) are minimizing geodesics such that $y_k := \gamma_k(0) \to y_\infty \in X_\infty$ and $a_k \to a_\infty \in [-\infty, 0]$ resp. $b_k \to b_\infty \in [0, \infty]$, then the γ_k subconverge to some minimizing geodesic $\gamma_\infty : [a_\infty, b_\infty] \to X_\infty$ with $\gamma_\infty(0) = y_\infty$ where we have to replace $[a_\infty, b_\infty]$ by $(-\infty, b_\infty]$, $[a_\infty, \infty)$ or $\mathbb{R}$ if $a_\infty = -\infty$ or $b_\infty = \infty$.

3.2 Smooth convergence

3.2.1 Definition

Let $((M_k, g_k), x_k)$ be a sequence of pointed complete n dimensional Riemannian manifolds. Assume that their path metric Gromov-Hausdorff converges to some metric space (X_∞, x_∞). It may happen that X_∞ or parts of X_∞ come from the structure of a smooth Riemannian manifold of the same dimension. So assume that there is some open set $M_\infty \subset X_\infty$ that is locally isometric to the path metric of a Riemannian manifold. Hence, M_∞ is itself a smooth Riemannian manifold (M_∞, g_∞). We now want to establish a stronger notion of convergence: Assume that the M_k and X_∞ are embedded into a metric space Z as described in (D′). We say that the $((M_k, g_k), x_k)$ *smoothly converge* to (X_∞, x_∞) on M_∞ in the above embedding if there are an exhaustion of M_∞ by open sets U_k and diffeomorphisms onto their image $\Phi_k : U_k \to M_k$ such that $\Phi_k \to \mathrm{id}_{M_\infty}$ pointwise and

$$\Phi_k^* g_k \to g_\infty \quad \text{smoothly for } k \to \infty.$$

Hereby we mean that we have pointwise convergence $\Phi_k^* g_k \to g_\infty$ and $\nabla^l \Phi_k^* g_k \to 0$ for all l, where ∇ denotes the Levi-Civita connection on (M_∞, g_∞). Observe that it may depend on the embeddings of the M_k and X_∞ into Z whether we have smooth convergence on M_∞ or not.

In order to quantify smooth closeness, we introduce the notion of *smooth ε-isometries*: We call a diffeomorphism onto its image $\Phi : (N, g_N) \to (M, g_M)$ a *smooth ε-isometry* if

$$\|\Phi^* g_M - g_N\|_{C^{[\varepsilon^{-1}]}} < \varepsilon$$

Notice the analogy to (B). If there is no chance of confusion, we simply call a smooth ε-isometry an ε-isometry. Moreover, we call Φ an *ε-homothety* if it is an ε-isometry between (N, g_N) and $(M, \lambda^{-2} g_N)$ for some $\lambda > 0$ which we call the *scaling factor of* Φ.

3.2.2 Regularity of the limit

We want to find conditions under which we can ensure local smoothness of the limit space X_∞ at certain points. Observe that intuitively there may occur two different processes leading to nonsmoothness in the limit:

In the first case, necks pinch down to diameter 0. As a consequence the dimension drops when passing to the limit. The standard example is the family of 2-tori $T^2 = S^1 \times S^1$ that shrink down in one factor but stay constant in the other. However, it is also possible that neckpinches occur just at certain areas. In all these cases the injectivity radius can not be uniformly bounded from below by a certain positive constant.

In the second case, nonsmooth edges or corners develop: For example we could produce a 2 dimensional cylinder with a lid or a cylinder capped by a round hemisphere as a limit. Observe that near the nonsmooth point $\|R\|$ resp. $\|\nabla R\|$ does not stay bounded in the approximating sequence.

This motivates the following definition: Consider the functions

$$\mathrm{inj}, \|R\|, \|\nabla R\|, \|\nabla^2 R\|, \ldots : \bigcup_k M_k \subset Z \longrightarrow [0, \infty).$$

Call a point $y \in X_\infty$ *regular* if there is a neighborhood $U \subset Z$ and constants $\rho > 0$ and $C_0, C_1, \ldots < \infty$ such that for large k we have $\mathrm{inj} \geq \rho$ and $\|\nabla^l R\| \leq C_l$ for all l on $U \cap M_k$. Let M_∞ be the set of all regular points.

We first mention a technical result, namely that we can bound the derivatives of the metric in exponential coordinates in terms of the derivatives of the curvature. A proof can be found in [LB2]. Note that this result can be improved in harmonic coordinates (see [DK] and [JK]). However, we will not need this fact here.

Lemma 3.2.1. *For any $r < \infty$, $l, n \in \mathbb{N}$ and any constants $D_0, \dots, D_l < \infty$ we can find constants $D'_0, \dots, D'_l < \infty$ such that the following holds:*
Let (M, g) be a Riemannian manifold and $x \in M$. Consider a starlike subset $U \subset T_x M$ (with center 0) on which the exponential map $\exp$ is defined and that lies in $B_r(0)$. Suppose that $\|\nabla^i R\| < D_i$ for all $0 \le i \le k$. Then $\|\mathrm{d}^i \exp^* g\| < D'_i$ on U for all $0 \le i \le k$.

Theorem 3.2.2. *M_∞ is locally isometric to a smooth Riemannian manifold.*

Proof. Let $y \in M_\infty$. Then we have bounds for inj, $\|\nabla^l R\|$ on some neighborhood U of y. It is enough to show that the metric d_∞ restricted to some neighborhood of y is isometric to the path metric of a Riemannian manifold. Let $y_k \in M_k \cap U$ (for large k) be a sequence with $y_k \to y$. Choose $0 < r < \min(\frac{\varrho}{2}, \frac{\pi\sqrt{n(n-1)}}{4\sqrt{C_0}})$ and such that $\overline{B}^{M_k}_{2r}(y_k) \subset U$ for large k. Then we have pointed Gromov-Hausdorff convergence

$$\left(\overline{B}^{M_k}_{2r}(y_k), y_k\right) \xrightarrow[k\to\infty]{} \left(\overline{B}^{X_\infty}_{2r}(y), y\right).$$

Let

$$\exp_k : \mathbb{R}^n \supset B_{2r}(0) \to B^{M_k}_{2r}(y_k), \quad 0 \mapsto y_k$$

be exponential maps ($(\mathrm{d}\exp_k)_0$ may be chosen arbitrarily) which are diffeomorphisms since $r < \mathrm{inj}(y_k)$. Set $h_k := \exp_k^* g_k$. By Lemma 3.2.1 we may bound the h_k and their derivatives in terms of the (C_l) uniformly on $B_{2r}(0)$ and in k. Applying Arzelà-Ascoli's theorem we get that after passing to a subsequence

$$h_k \longrightarrow h_\infty \quad \text{smoothly} \tag{3.1}$$

for some smooth $h_\infty \in \mathrm{Sym}_2 T^* B_{2r}(0)$. h_∞ is a Riemannian metric since $2r < \frac{\pi\sqrt{n(n-1)}}{2\sqrt{C_0}}$, so we can exclude conjugate points for 0.

This result implies that the $\exp_k : (B_{2r}(0), h_\infty) \to (B_{2r}(y_k), g_k)$ are uniformly lipschitz. So again (by Arzelà-Ascoli) after passing to a subsequence we get

$$\exp_k \longrightarrow \exp_\infty \quad \text{uniformly} \tag{3.2}$$

for some $\exp_\infty : B_{2r}(0) \to B^{M_\infty}_{2r}(y)$.

It is easy to see that $d'_k := \left(\exp_k|_{B_r(0)}\right)^* d_k$ is equal to the restriction of the path metric of $(B_{2r}(0), h_k)$ to $B_r(0)$. From (3.1) we get that d'_k converges to the restriction of the path metric of $(B_{2r}(0), h_\infty)$ to $B_r(0)$. On the other hand, we conclude from (3.2) that d'_k converges to $d'_\infty := \left(\exp_\infty|_{B_r(0)}\right)^* d_\infty$ on $B_r(0)$. This gives us the desired result. □

Now define $\Phi_k : B^{M_\infty}_r(y) \to B^{M_k}_r(y_k)$ by

$$\Phi_k := \left(\exp_k|_{B_r(0)}\right) \circ \left(\exp_\infty|_{B_r(0)}\right)^{-1}.$$

We have $\Phi_k(y) = y_k$. From (3.2) we get that $\Phi_k \to \mathrm{id}_{B^{M_\infty}_r(y)}$ uniformly. Furthermore, by the Lemma at the end of this subsection (applied to $B^{M_\infty}_{2r}(0)$) we conclude that $\Phi_k^* g_k \to g_\infty|_{B^{M_\infty}_r(y)}$ in the smooth uniform sense

It is important to observe that although every regular point in X_∞ is smooth, a smooth point in X_∞ does not necessarily have to be regular.

Lemma 3.2.3. *Let $E \to M$ be a vector bundle with two connections ∇ and $\widetilde{\nabla}$. If a sequence of sections $s_k \in \Gamma E$ smoothly converges to some section $s_\infty \in \Gamma E$ with respect to ∇, i.e.*

$$\nabla^l s_k \longrightarrow \nabla^l s_\infty \qquad \text{in } (T^*M)^{\otimes l} \otimes E \text{ for all } l \ge 0$$

then it also converges smoothly with respect to $\widetilde{\nabla}$.

Proof. Without loss of generality we may assume that $s_\infty = 0$. Let

$$A := \widetilde{\nabla} - \nabla \in \Gamma(T^*M \otimes \operatorname{End} E)$$

We want to show

$$\widetilde{\nabla}^l s_k \longrightarrow 0.$$

We proceed by induction on l. For $l = 0$ there is nothing to show. Let the hypothesis be true for l. We will show that it also holds for $l+1$. Obviously, we have $\nabla s_k \to 0$ smoothly in $\Gamma T^*M \otimes E$ with respect to ∇. So by the induction hypothesis we get

$$\widetilde{\nabla}^l \nabla s_k \longrightarrow 0.$$

Again by the induction hypothesis we have

$$\widetilde{\nabla}^l_{v_1,\dots,v_l}\left(\widetilde{\nabla} s_k - \nabla s_k\right) = \widetilde{\nabla}^l_{v_1,\dots,v_l} A(s_k) = \sum_{i=0}^{l}\sum_{\sigma\in S_l} \widetilde{\nabla}^i_{v_{\sigma 1},\dots,v_{\sigma i}} A(\widetilde{\nabla}^{l-i}_{v_{\sigma(i+1)},\dots,v_{\sigma l}} s_k) \longrightarrow 0.$$

Combining the two convergence results we get $\widetilde{\nabla}^{l+1} s_k \longrightarrow 0$. ☐

3.2.3 Establishing smooth convergence

Now we want to show that we have smooth convergence on M_∞ for a subsequence of the (M_k). Choose an exhaustion of M_∞ by open and in M_∞ relatively compact sets $U_l \subset M_\infty$. It suffices to show that for any l there is a subsequence such that there are smooth maps $\Phi_k : U_l \to M_k$ with the property that $\Phi_k \to \mathrm{id}_{U_l}$ pointwise and $\Phi_k^* g_k \to g_\infty|_{U_l}$ smoothly. An application of a diagonal argument yields the desired result. For this reason we will discard the index l in the following steps.

The result after Theorem 3.2.2 implies that after passing to a subsequence there is an $r > 0$ (less than the injectivity radius of M_∞ on U) and a finite number of points $y^{(1)}, \dots, y^{(m)} \in U$ that represent an $\frac{r}{64}$-net for $\overline{U}$ as well as diffeomorphisms

$$\Phi_k^{(i)} : B_r^{M_\infty}(y^{(i)}) \longrightarrow B_r^{M_k}(y_k^{(i)}) \qquad \text{where} \qquad y_k^{(i)} := \Phi_k(y^{(i)})$$

such that $\Phi_k^{(i)} \to \mathrm{id}_{B_r^{M_\infty}(y^{(i)})}$ uniformly and $(\Phi_k^{(i)})^* g_k \to g_\infty|_{B_r^{M_\infty}(y^{(i)})}$ uniformly smoothly. For two indices $i, j \in \{1, \dots, m\}$ write $i \prec j$ if $\overline{B}_{r/4}^{M_\infty}(y^{(i)}) \subset B_r^{M_\infty}(y^{(j)})$. Then, for sufficiently large k, we also have $\overline{B}_{r/4}^{M_k}(y_k^{(i)}) \subset B_r^{M_k}(y_k^{(j)})$. By passing to a subsequence we may assume that this is always implied by $i \prec j$.

For arbitrary i, j we have $d_k(\Phi_k^{(i)}, \Phi_k^{(j)}) \to 0$ uniformly on $B_r^{M_\infty}(y^{(i)}) \cap B_r^{M_\infty}(y^{(j)})$. In order to express an analogous statement for the derivatives of $\Phi_k^{(i)}$ resp. $\Phi_k^{(j)}$, we define

$$\Psi_k^{(ji)} : B_{r/4}^{M_\infty}(y^{(i)}) \longrightarrow B_r^{M_\infty}(y^{(j)}) \qquad \text{by} \qquad \Psi_k^{(ji)} := \left(\Phi_k^{(j)}\right)^{-1} \circ \Phi_k^{(i)}$$

if $i \prec j$. The previous result gives us that

$$\Psi_k^{(ji)} \longrightarrow \mathrm{id}_{B_{r/4}^{M_\infty}(y^{(i)})} \qquad \text{uniformly.}$$

We wish to make this a smooth convergence by choosing a suitable subsequence.

The differential $(\mathrm{d}\Psi_k^{(ji)})_{y^{(i)}} : T_{y^{(i)}} M_\infty \to T_{\Psi_k^{(ji)}(y^{(i)})} M_\infty$ is uniformly bounded in k with respect to h_∞ since it is orthogonal with respect to the metric $(\Phi_k^{(i)})^* g_k$ which is close to h_∞. This and the fact that $\Psi_k^{(ji)}(y^{(i)}) \to y^{(i)}$ implies that we can choose a subsequence such that

$$\left(\mathrm{d}\Psi_k^{(ji)}\right)_{y^{(i)}} \longrightarrow \varphi^{(ji)} \qquad \text{for all } i \prec j$$

where $\varphi^{(ji)} : (T_{y^{(i)}}M_\infty, (h_\infty)_{y^{(i)}}) \to (T_{y^{(i)}}M_\infty, (h_\infty)_{y^{(i)}})$ is some isometry. We are in the following situation:

$$\begin{tikzcd}
B^{M_\infty}_{r/4}(y^{(i)}) \arrow[d, "\Psi_k^{(ji)}"'] & B_{r/4}(0) \arrow[l, "\exp_{y^{(i)}}"'] \arrow[r, "\exp_{y_k^{(i)}}"] \arrow[dl, dashed, "\exp_k^{(ji)} := \exp^{(\Phi_k^{(j)})^* g_k}_{\Psi_k^{(ji)}(y^{(i)})}"] & B^{M_k}_{r/4}(y_k^{(i)}) \arrow[d, "\cap"] \\
B^{M_\infty}_r(y^{(j)}) \arrow[rr, "\Phi_k^{(j)}"'] & & B^{M_k}_r(y_k^{(j)})
\end{tikzcd}$$

where $\exp_k^{(ji)}$ is the exponential map on $B_r^{M_\infty}$ equipped with the metric $(\Phi_k^{(j)})^* g_k$. We have $\exp_k^{(ji)}(0) = \Psi_k^{(ji)}(y^{(i)}) \to y^{(i)}$ and

$$\left(\mathrm{d}\exp_k^{(ji)}\right)_0 = \left(\mathrm{d}\Psi_k^{(ji)}\right)_{y^{(i)}} \circ \left(\mathrm{d}\exp_{y^{(i)}}\right)_0 \longrightarrow \varphi^{(ji)} \circ \left(\mathrm{d}\exp_{y^{(i)}}\right)_0 .$$

Since $(\Phi_k^{(j)})^* g_k \to g_\infty$ smoothly, we have smooth uniform convergence

$$\exp_k^{(ji)} \longrightarrow \exp^{(ji)}$$

where $\exp^{(ji)} : B_{r/4}(0) \to B_r^{M_\infty}(y^{(j)})$ is an exponential map with $\exp^{(ji)}(0) = y^{(i)}$ and $(\mathrm{d}\exp^{(ji)})_0 = \varphi^{(ji)} \circ (\mathrm{d}\exp_{y^i})_0$. Since $\Psi_k^{(ji)}$ converges uniformly to $\mathrm{id}_{B^{M_\infty}_{r/4}(y^i)}$ we get $\exp^{ji} = \exp_{y^{(i)}}$. So we must have $\varphi^{(ji)} = \mathrm{id}_{T_{y^{(i)}}M}$ and thus

$$\Psi_k^{(ji)} \longrightarrow \mathrm{id}_{B^{M_\infty}_{r/4}(y^{(i)})} \qquad \text{smoothly uniformly.}$$

We have found out that the maps $\Phi_k^{(i)}|_{B^{M_\infty}_{r/4}(y^{(i)}}$ get arbitrarily close to each other on their overlaps even in the smooth sense. Now, we will interpolate between them (actually we will interpolate between their inverses).

Let $V_k := \bigcup_i \Phi_k^{(i)}(B^{M_\infty}_{r/4}(y^{(i)})) = \bigcup_i B^{M_k}_{r/4}(y_k^{(i)}) \subset M_k$ and $V := \bigcup_i B^{M_\infty}_{r/4}(y^{(i)}) \subset M_\infty$. Assume that we have constructed maps $\Lambda_k : V_k \to M_\infty$ such that

$$\Lambda_k \circ \Phi_k^{(i)}|_{B^{M_\infty}_{r/4}(y^{(i)})} \longrightarrow \mathrm{id}_{B^{M_\infty}_{r/4}(y^{(i)})} \qquad \text{smoothly uniformly} \tag{3.3}$$

for each i. We will show that this gives us the desired result. Since for large k the map $\Lambda_k \circ \Phi_k^{(i)}|_{B^{M_\infty}_{r/16}(y^{(i)})}$ is bilipschitz with a Lipschitz constant that is arbitrarily close to 1, we know that its image contains $B^{M_\infty}_{r/64}(y^{(i)})$ for large k. So for large k we have $\Lambda_k(V_k') \supset U$ for $V_k' := \bigcup_i B^{M_k}_{r/16}(y^{(i)})$.

Furthermore, we can assume that the $\Lambda_k' := \Lambda_k|_{V_k'}$ are injective for large k for the following reason: If k is large we can assume injectivity for each $\Lambda_k|_{B^{M_k}_{r/4}(y_k^{(i)})}$. Suppose there are two points $z_1, z_2 \in V_k'$ with $\Lambda_k(z_1) = \Lambda_k(z_2)$. Then for large k the points z_1 or z_2 never lie in the same $B^{M_k}_{r/4}(y_k^{(i)})$, hence $d_k(z_1, z_2) \geq \frac{3}{16} r$. But because of (3.3) the Λ_k converge pointwise to id_V. So we get a contradiction for large k.

We have deduced that $\Lambda_k' : V_k' \to \Lambda_k'(V_k')$ is invertible and $(\Lambda_k \circ \Phi_k^{(i)})(B^{M_\infty}_{r/16}(y^{(i)})) \supset B^{M_\infty}_{r/64}(y^{(i)})$ for large k. From (3.3) we conclude that

$$\left(\Lambda_k \circ \Phi_k^{(i)}\right)^{-1}\Big|_{B^{M_\infty}_{r/64}(y^{(i)})} \longrightarrow \mathrm{id}_{B^{M_\infty}_{r/64}(y^{(i)})}$$

smoothly uniformly for any i and thus we have smooth convergence

$$\left((\Lambda_k')^{-1}\right)^* g_k\Big|_{B^{M_\infty}_{r/64}(y^{(i)})} = \underbrace{\left((\Lambda_k' \circ \Phi_k^{(i)})^{-1}\Big|_{B^{M_\infty}_{r/64}(y^{(i)})}\right)^*}_{\to \mathrm{id}_{B^{M_\infty}_{r/64}(y^{(i)})}} \underbrace{\left(\Phi_k^{(i)}\right)^* g_k}_{\to g_\infty} \longrightarrow g_\infty$$

for any i. Moreover, since $\Phi_k^{(i)} \to \Phi^{(i)}$ for each i we get that $(\Lambda_k')^{-1} \to \mathrm{id}_U$ uniformly on U. Now set $\Lambda_k'' := \Lambda_k'|_{(\Lambda')^{-1}(U)} : (\Lambda')^{-1}(U) \to U$. This map is invertible and we have $((\Lambda_k'')^{-1})^* g_k \to g_\infty$ smoothly uniformly and $(\Lambda_k'')^{-1} \to \mathrm{id}_U$ uniformly.

This proves the hypothesis.

It remains to construct the Λ_k. Let $\widetilde{\Delta} \subset M^m$ be an open neighborhood of the diagonal $\Delta \subset M^m$. We call a function

$$\sum : \{(s_1, \ldots, s_m) \in [0,1]^m : s_1 + \ldots + s_m = 1\} \times \widetilde{\Delta} \longrightarrow M_\infty$$

an *averaging function* if the following holds (we will write $\sum s_i z_i$ instead of $\sum(s_1, \ldots, s_m, z_1, \ldots, z_m)$):

(i) $\sum s_i z_i = \sum s_i z_i'$ if $s_i = 0$ or $z_i = z_i'$ for each i,
(ii) $\sum s_i z_i = z_j$ if $s_i = \delta_{ij}$ and
(iii) $\sum s_i z = z$.

Clearly such a function exists. Fix an averaging function $\sum$ and choose a partition of unity (λ_i) on $(\overline{B}^{M_\infty}_{r/4}(y^{(i)}))$. Define

$$\Lambda_k := \sum \frac{\lambda_i \circ (\Phi_k^{(i)})^{-1}}{\sum \lambda_s \circ (\Phi_k^{(s)})^{-1}} \left(\Phi_k^{(i)}\right)^{-1}\Bigg|_{V_k}$$

Since $\Psi_k^{(ji)} = (\Phi_k^{(j)})^{-1} \circ \Phi_k^{(i)}|_{B^{M_\infty}_{r/4}} \to \mathrm{id}_{B^{M_\infty}_{r/4}}$ if $i \prec j$ and $\lambda_j \circ (\Phi_k^{(j)})^{-1} \to 0$ on $B^{M_k}_{r/4}(y_k^{(i)})$ if $i \not\prec j$, we get (3.3). This completes the proof.

Finally, we mention a special case of what we have proven so far.

Theorem 3.2.4. *Let (M_k, x_k) be a sequence of pointed complete n dimensional manifolds that satisfy* $\mathrm{inj} \geq \rho$ *and $\|\nabla^l R\| \leq C_l$ for all $l \geq 0$. Then there is a subsequence (M_{k_i}, x_{k_i}) that smoothly converges to some pointed complete n dimensional Riemannian manifold (M_∞, x_∞).*

Observe that from Proposition 3.4.1 and 3.4.2 we can deduce that it already suffices to assume $\mathrm{inj}(x_k) \geq \rho$ instead of $\mathrm{inj} \geq \rho$.

3.3 Compactness of Ricci flows

Let $(M_k, (g_0)_k)$ be a sequence of complete Riemannian manifolds and $x_k \in M_k$. Assume that we have Gromov-Hausdorff convergence of the M_k resp. (M_k, x_k) (if not all M_k are compact) to a metric space X_∞ such that the convergence is smooth on M_∞. Fix this convergence, e.g. by embedding the spaces M_k and X_∞ into an ambient metric space Z as in (D) resp. (D′) in subsection 3.1.1 resp. 3.1.2. Let g_∞ be the metric on M_∞ and $M'_\infty \subset M_\infty$ be open. Consider Ricci flows $(g_t)_k$ on $M_k \times I_k$ (we claim $0 \in I_k$) such that $(g_0)_k = g_k$. Let I_∞ be an interval (open, half open or closed) and assume that the endpoints of I_k converge to the endpoints of I_∞. We say that the Ricci flows $(g_t)_k$ *converge to a Ricci flow* $(g_t)_\infty$ on $M'_\infty \times I_\infty$ if there are an exhaustion U_k of M'_∞ of open sets, an ascending sequence of intervals $J_1 \subset J_2 \subset \cdots$ whose union is I_∞ and diffeomorphisms $\Phi_k : U_k \to M_k$ such that $\Phi_k \to \mathrm{id}_{M'_\infty}$ pointwise and $(\Phi_k^*(g_\cdot)_k)|_{U_k \times J_k} \to (g_\cdot)_\infty$ smoothly for $k \to \infty$.

We will now prove a compactness result for Ricci flows. Consider a sequence of Ricci flows $M_k \times I_k$ with complete time slices and assume that the endpoints of I_k converge to the endpoints of some interval $I_\infty = (T_1^\infty, T_2^\infty)$ or $(T_1^\infty, T_2^\infty]$ if $T_2^\infty \in I_k$ for large k. Furthermore, we claim that $T_1^\infty < 0$. Let $x_k \in M_k$. Assume that for the time 0 slices $M_k(0)$ we have Gromov-Hausdorff convergence

$$(M_k(0), x_k) \xrightarrow[k \to \infty]{} (X_\infty, x_\infty) \tag{3.4}$$

where (X_∞, x_∞) denotes some pointed metric space. Imagine the M_k and X_∞ to be embedded into an ambient space Z. If the M_k are compact we can discard the basepoints. Note that (3.4) always holds for a subsequence if there is a uniform lower bound for Ric_k that may depend on the distance to x_k.

Let $M'_\infty \subset X_\infty$ be open. Assume that for any point $x \in M'_\infty$ and any compact interval $K \subset I_\infty$ there is a neighborhood $U \subset Z$ around x and constants $C_0 < \infty$, $\rho > 0$ such that $\|R\| \leq C_0$ on $(U \cap M_k) \times K$ and $\mathrm{inj} \geq \rho$ on $U \cap M_k$ for large k. Observe that we do not control higher curvature derivatives.

Using Shi's estimates, we conclude that for any $x \in M'_\infty$ and any compact interval $K \subset I_\infty$ there are a neighborhood $U \subset Z$ and constants $C_1, C_2, \ldots < \infty$ such that $\|\nabla^l R\| \leq C_l$ on $(U \cap M_k) \times K$ for large k. By the result of section 3.2 we conclude that after passing to a subsequence the convergence (3.4) is smooth on M'_∞. Let g_∞ be the limit metric.

We will now show that after passing to a subsequence the Ricci flows $M \times I_k$ converge to a Ricci flow $M'_\infty \times I_\infty$ on M'_∞. By the last paragraph there are an exhaustion U_k of M'_∞ and diffeomorphisms $\Phi_k : U_k \to M_k$ such that $\Phi_k \to \mathrm{id}_{M'_\infty}$ pointwise and $\Phi_k^* g_k \to g_\infty$ smoothly for $k \to \infty$. We can assume the U_k to be relatively compact. Observe that for any j and any compact interval $K \subset I_\infty$ there are constants $C_1, C_2, \ldots < \infty$ such that for all $l \geq 0$ we have $\|\Phi_k^*(\nabla^l R)\| \leq C_l$ on $U_j \times K$ for large k. Let $|\cdot|$ be the norm and $\tilde{\nabla}$ the Levi-Civita connection coming from g_∞. Using Lemma 3.3.2 at the end of this section, we conclude that for any j, l and any compact interval $K \subset I_\infty$ there is a uniform bound for $|\tilde{\nabla}^l (g_\cdot)_k|$ on $U_j \times K$ for large k. By the Ricci flow equation these bounds imply bounds on derivatives of $(g_t)_k$ that also involve the time direction. We can now use Arzelà-Ascoli's Theorem to conclude that there is a subsequence such that $\Phi_k^*(g_\cdot)_k$ smoothly converges to some limit flow $(g_\cdot)_\infty$ on $U_i \times K$. By a diagonal argument we get that after passing to a subsequence we have smooth convergence of the $\Phi_k^*(g_\cdot)_k$ to a limit flow $(g_\cdot)_\infty$ on $M_\infty \times I_\infty$. Obviously $(g_0)_\infty = g_\infty$. This completes the proof.

As a special case we mention the following Theorem

Theorem 3.3.1. *Let $M \times I_k$ be a sequence of Ricci flows with complete time slices and $x_k \in M$. Assume that the endpoints of I_k converge to the endpoints of some interval $I_\infty = (T_1^\infty, T_2^\infty)$ or $(T_1^\infty, T_2^\infty]$ if $T_2^\infty \in I_k$ for large k. Moreover, assume that $T_1^\infty < 0$ and that we have a uniform lower bound $\rho > 0$ for inj_0 (i.e. the injectivity radius of $M(0)$) and an upper bound for $\|R\|$ on all $M_k \times I_k$.*

Then there is a subsequence $(M_{k_i}(0), x_{k_i})$ that Gromov-Hausdorff converges to a pointed complete Riemannian manifold $(M_\infty(0), x_\infty)$ that is smooth everywhere and the Ricci flows $M \times I_k$ converge to a Ricci flow $M_\infty \times I_\infty$ on M_∞.

Observe again that by the remarks after Theorem 3.2.4 it suffices to assume that $\mathrm{inj}_0(x_k) \geq \rho$ rather than $\mathrm{inj}_0 \geq \rho$. In the situation of the Theorem we also say that the sequence $(M_{k_i} \times I_{k_i}, (x_{k_i}, 0))$ of pointed Ricci flows converges smoothly to $(M_\infty \times I_\infty, (x_\infty, 0))$.

Lateron we will sometimes say that a sequence $(M_k \times I_k, (x_k, t_k))$ of pointed Ricci flows smoothly converges to some Ricci flow $(M_\infty \times I_\infty, (x_\infty, t_\infty))$ if we have convergence of the Ricci flows that are shifted in time by $-t_k$ resp. $-t_\infty$.

Lemma 3.3.2. *Let $l \in \mathbb{N}$. For any $-\infty < T_1 \leq 0$, $0 \leq T_2 < \infty$ and $C', C_0, \ldots, C_l < \infty$ we find constants $D_0, \ldots, D_l < \infty$ such that the following statement holds:*
Let $M \times [T_1, T_2]$ be a Ricci flow with $\|\nabla^i R\| < C_i$ everywhere for $0 \leq i \leq l$. Consider a metric $\tilde{g}$ on M that is C'-bilipschitz to g_0 and denote by $|\cdot| := \|\cdot\|_0$ and $\tilde{\nabla}$ the induced norm resp. Levi-Civita connection. Then $|\tilde{\nabla}^i g_t| < C_i$ for all $t \in [T_1, T_2]$ and $0 \leq i \leq l$.

Proof. We first prove that the norms $\langle \cdot, \cdot \rangle_t$ are D'-Bilipschitz for a universal constant D'. For any $v \in TM$ we have
$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_t^2 = -2\operatorname{Ric}_t(v,v)$$
and $|\operatorname{Ric}_t(v,v)| \leq \|\operatorname{Ric}_t\|_t \|v\|_t^2 \leq \sqrt{n-1}\|R_t\|_t\|v\|_t^2$. So
$$-2\sqrt{n-1}C_0\|v\|_t^2 \leq \frac{\mathrm{d}}{\mathrm{d}t}\|v\|_t^2 \leq 2\sqrt{n-1}C_0\|v\|_t^2.$$
Hence $D' = C' \max\{\exp(-2\sqrt{n-1}C_0T_1), \exp(2\sqrt{n-1}C_0T_2)\}$ is an appropriate constant.

Next, we prove a refined version of Lemma 3.2.3. We claim that any tensor $s \in \Gamma T^p_q M$ satisfies
$$\|\tilde{\nabla}^k s - \nabla^k s\| \leq E \sum_{\substack{i_0 + \ldots + i_m = k, \\ i_0 < k,\ m \geq 0}} \|\tilde{\nabla}^{i_0} s\| \|\tilde{\nabla}^{i_1} g\| \cdots \|\tilde{\nabla}^{i_m} g\| \tag{3.5}$$
at any time $t \in [T_1, T_2]$ where E depends only on n, p, q and k (g denotes the metric and $\|\cdot\|$ the norm at time t). The proof is by induction. For $k = 0$ there is nothing to show. Assume that (3.5) holds for k. Then after replacing s by $\tilde{\nabla} s$ we get
$$\|\tilde{\nabla}^{k+1} s - \nabla^k \tilde{\nabla} s\| \leq E_1 \sum_{\substack{i_0 + \ldots + i_m = k, \\ i_0 < k,\ m \geq 0}} \|\tilde{\nabla}^{i_0+1} s\| \|\tilde{\nabla}^{i_1} g\| \cdots \|\tilde{\nabla}^{i_m} g\|. \tag{3.6}$$
Moreover, the Koszul formula implies
$$\tilde{\nabla} s - \nabla s = \tilde{\nabla} g *_g s =: s'.$$
Here $\tilde{\nabla} g *_g s$ denotes an $O(g)$ equivariant contraction of $\tilde{\nabla} g \otimes s$. An application of (3.5) for s' gives
$$\|\nabla^k s'\| \leq \|\tilde{\nabla}^k s'\| + E_2 \sum_{\substack{i_0 + \ldots + i_m = k, \\ i_0 < k,\ m \geq 0}} \|\tilde{\nabla}^{i_0} s'\| \|\tilde{\nabla}^{i_1} g\| \cdots \|\tilde{\nabla}^{i_m} g\|. \tag{3.7}$$
Observe that
$$\|\tilde{\nabla}^i s'\| = \|\tilde{\nabla}^i(\tilde{\nabla} g *_g s)\| = E_3 \sum_{\substack{i_0 + \ldots + i_m = i+1, \\ i_0 < i+1,\ m \geq 0}} \|\tilde{\nabla}^{i_0} s\| \|\tilde{\nabla}^{i_1} g\| \cdots \|\tilde{\nabla}^{i_m} g\|. \tag{3.8}$$
Combining (3.6), (3.7) and (3.8) gives
$$\begin{aligned} \|\tilde{\nabla}^{k+1} s - \nabla^{k+1} s\| &\leq \|\tilde{\nabla}^{k+1} s - \nabla^k \tilde{\nabla} s\| + \|\nabla^k \tilde{\nabla} s - \nabla^{k+1} s\| \\ &\leq E_4 \sum_{\substack{i_0 + \ldots + i_m = k+1, \\ i_0 < k+1,\ m \geq 0}} \|\tilde{\nabla}^{i_0} s\| \|\tilde{\nabla}^{i_1} g\| \cdots \|\tilde{\nabla}^{i_m} g\| \end{aligned}$$

as desired.

In the first part of the proof we showed that the assertion holds for $l = 0$. Now assume that it is true for l. We will show that it also holds for $l + 1$. First observe that by successive application of (3.5) we can show that there are constants $F_0, \ldots, F_l < \infty$ that only depend on $n, C_1, \ldots, C_l, D_1, \ldots, D_l$ such that $\|\tilde{\nabla}^j \operatorname{Ric}_t\| < F_j$ for all $0 \leq j \leq l$. Next, we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\tilde{\nabla}^{l+1} g_t\right|^2 = 2\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\nabla}^{l+1} g_t, g_t\right\rangle_0 = -4\left\langle \tilde{\nabla}^{l+1}\operatorname{Ric}_t, g_t\right\rangle_0$$

and

$$\begin{aligned}\left|\left\langle \tilde{\nabla}^{l+1}\operatorname{Ric}_t, g_t\right\rangle_0\right| &\leq \left|\tilde{\nabla}^{l+1}\operatorname{Ric}_t\right| \cdot |g_t| \leq \sqrt{D'}\left\|\tilde{\nabla}^{l+1}\operatorname{Ric}_t\right\|_t \cdot |g_t| \\ &\leq \sqrt{D'}\|\nabla^{l+1}\operatorname{Ric}_t\|_t \cdot |g_t| + \sqrt{D'}E \sum_{\substack{i_0+\ldots+i_m = l+1,\\ i_0 < l+1,\, m \geq 0}} \|\tilde{\nabla}^{i_0}\operatorname{Ric}_t\|_t \|\tilde{\nabla}^{i_1} g_t\|_t \cdots \|\tilde{\nabla}^{i_m} g_t\|_t \\ &\leq \sqrt{nD'}D'C_{l+1} + \sqrt{D'}E \sum_{\substack{i_0+\ldots+i_m = l+1,\\ i_0 < l+1,\, m \geq 0}} (D')^{(i_1/2+1)+\ldots+(i_m/2+1)} F_{i_0} |\tilde{\nabla}^{i_1} g_t| \cdots |\tilde{\nabla}^{i_m} g_t|\end{aligned}$$

We can bound all terms on the right hand side except $|\tilde{\nabla}^{l+1} g_t|$. Thus

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left|\tilde{\nabla}^{l+1} g_t\right|^2\right| \leq A|\tilde{\nabla}^{l+1} g_t| + B|\tilde{\nabla}^{l+1} g_t|^2$$

for some constants A, B. Integrating this inequality gives the desired result. □

3.4 A lower bound on the injectivity radius

We now introduce a tool that will later help us to bound injectivity radii by more convienient quantities.

Proposition 3.4.1. *For any $n \in \mathbb{N}$, $\kappa, r < \infty$ and $\mu > 0$ there is a constant $\alpha(n, \kappa r^2, \frac{\mu}{r^n}) > 0$ such that the following holds:*
Let (M, g) be a Riemannian manifold, $x \in M$ such that $B_r(x)$ is relatively compact and $-(n-1)\kappa \leq \operatorname{Ric}$ as well as for the sectional curvature $K \leq \kappa$ on $B_r(x)$. Then if $\operatorname{vol} B_r(x) \geq \mu$, we have $\operatorname{inj}(x) \geq r\alpha$.

Proof. See [CGT, Thm 4.3]. Observe that the lower sectional curvature bound can be replaced by the lower bound on the Ricci curvature without modifications of the proof. Moreover, by the Bishop-Gromov Theorem 1.4.2 we may replace the lower bound on the volume of a smaller ball $B_s(p) \subset B_r(p)$ by a lower bound on the volume of the ball $B_r(x)$ itself. □

The reverse can be shown easily

Proposition 3.4.2. *For any $n \in \mathbb{N}$, $\rho > 0$ and $\kappa < \infty$ there is a constant $\mu'(n, \kappa\rho^2)$ such that if (M, g) is a Riemannian manifold, $\operatorname{inj}(x) \geq \rho$ for some $x \in M$ and the sectional curvature on $B_\rho(x)$ satisfies $K \leq \kappa$, then $\operatorname{vol} B_\rho(x) > \mu'\rho^n$.*

3.5 Tangential cones

Proposition 3.5.1. *Let (X, d_X) be a metric space that is locally Alexandrov of curvature ≥ 0 in $p \in X$. Assume that for any $\varepsilon > 0$ the quantity $N_{\lambda\varepsilon}^{(X,p)}(\lambda)$ is uniformly bounded for $\lambda \to 0$. Denote by $\frac{1}{\lambda}X$ the rescaled space $(X, \frac{1}{\lambda}d_X)$. Then $(\frac{1}{\lambda}X, p)$ Gromov-Hausdorff converges as $\lambda \to 0$ to a metric cone (C, p_∞) which an Alexandrov space of curvature ≥ 0.*

We call (C, p_∞) the *tangential cone* of X in p. Note that the Proposition is still true if we just require (X, d) to be Alexandrov of curvature $\geq \kappa$ in p. However, we will not need this fact subsequently.

Proof. We just consider the case in which (X, d) is *globally* an Alexandrov space of curvature ≥ 0. It will be clear how to manage the local case. In the following we will consider all minimizing geodesics to be parameterized by arclength. Consider the quotient

$$N := \{\sigma : [0, \delta] \to X \ \ : \ \ \delta > 0, \ \ \sigma \text{ minimizing geodesic}, \ \ \sigma(0) = p\}/\sim$$

where $\sigma_1 \sim \sigma_2$ if $\sigma_1|_{[0,\delta_2]} = \sigma_2$ or $\sigma_1 = \sigma_2|_{[0,\delta_1]}$ for two minimizing geodesics $\sigma_{1/2} : [0, \delta_{1/2}] \to X$ (recall that geodesics in Alexandrov spaces do not branch). Define a metric on N via

$$d_N([\sigma_1], [\sigma_2]) := \sphericalangle_p(\sigma_1, \sigma_2) = \lim_{t,s\to 0} \widetilde{\sphericalangle}\sigma_1(s_1)p\sigma_2(s_2).$$

Observe that by property (B) in the definition of Alexandrov spaces, this limit exists and is positive if $[\sigma_1] \neq [\sigma_2]$. Moreover, $\operatorname{diam} N \leq \pi$.

We show that N is totally bounded: Assume not. Then for some $\varepsilon > 0$ there is sequence $[\sigma_1], [\sigma_2], \ldots \in N$ (where the $\sigma_i : [0, \delta_i] \to X$ are minimizing geodesics starting in p) such that $d_N(\sigma_i, \sigma_j) \geq \varepsilon$ for all $i \neq j$. This implies that for $\varepsilon' = \sin(\frac{\varepsilon}{2})$ any two balls $B_{\lambda\varepsilon'}(\sigma_i(\lambda))$ and $B_{\lambda\varepsilon'}(\sigma_j(\lambda))$ are disjoint for $i \neq j$ and $\lambda \in [0, \delta_i] \cap [0, \delta_j]$. Now pick an arbitrary number $k \in \mathbb{N}$. Choose $\lambda > 0$ so small that $\lambda < \delta_1, \ldots, \delta_k$. Then the balls $B_{\lambda\varepsilon'}(\sigma_i(\lambda))$ are pairwise disjoint for $i = 1, \ldots, k$. Choose a $\lambda\varepsilon'$-net $\{x_1, \ldots, x_m\}$ of $B_\lambda(p)$. It is easy to see that $m \geq k$ since otherwise there would be two points $\sigma_i(\lambda)$, $\sigma_j(\lambda)$ that lie within a $\lambda\varepsilon'$ distance to some x_l contradicting the disjointness mentioned above. So for any k we have $N^{(X,p)}_{\lambda\varepsilon'}(\lambda) \geq k$ for small λ contradicting the assumption of the Proposition.

Let $\overline{N}$ be the completion of N. Obviously $\overline{N}$ is compact. Denote by (C, p_∞) the metric cone over $(\overline{N}, d_{\overline{N}})$. It is clear that

$$\begin{aligned} d_C\left((t_1, [\sigma_1]), (t_1, [\sigma_1])\right) &= \sqrt{t_1^2 + t_2^2 - 2t_1t_2 \cos \lim_{\lambda\to 0} \widetilde{\sphericalangle}\sigma_1(\lambda t_1)p\sigma_2(\lambda t_2)} \\ &= \lim_{\lambda\to 0} \frac{1}{\lambda} d_X\left(\sigma_2(\lambda t_1), \sigma_2(\lambda t_2)\right) \end{aligned} \tag{3.9}$$

Recall that $\frac{1}{\lambda} d_X\left(\sigma_2(\lambda t_1), \sigma_2(\lambda t_2)\right)$ is nonincreasing in λ.

We will now show that we have Gromov-Hausdorff convergence of the $(\frac{1}{\lambda}X, p)$ to (C, p_∞) for $\lambda \to 0$. For this we will check property (A$'$) of subsection 3.1.2. Let $\varepsilon > 0$ and $r < \infty$. Choose an ε-net $\{x_0 = p_\infty, x_1, \ldots, x_m\}$ of $B^C_r(p_\infty)$ where $x_i = (t_i, [\sigma_i])$. Let $0 < \lambda \leq 1$ be so small that we can assume all σ_i to be defined on $[0, \lambda]$ and set $x_i^\lambda := \sigma_i(\lambda t_i)$.

We first show that $\{x_0^\lambda = p, x_1^\lambda, \ldots, x_m^\lambda\}$ is an ε-net of $B^{\frac{1}{\lambda}X}_r(p)$ or equivalently a $\lambda\varepsilon$-net of $B^X_{\lambda r}(p)$: Let $y \in B^X_{\lambda r}(p)$ and choose a minimizing geodesic $\gamma : [0, l] \to X$ between p and y. Then $y' := (l, [\gamma]) \in B^C_r(p_\infty)$ and thus $d_C(y', x_i) < \varepsilon$ for some $0 \leq i \leq m$. By (3.9) we have $d_X(\gamma(\lambda l), \sigma_i(\lambda t_i)) \leq \lambda d_C(y', x_i) < \lambda\varepsilon$.

Furthermore, (3.9) implies that for small λ we have

$$\left| \frac{1}{\lambda} d_X(x_i^\lambda, x_j^\lambda) - d_C(x_i, x_j) \right| < \varepsilon \qquad \text{for any } 0 \leq i, j \leq m.$$

Using property (C) in the Definition of Alexandrov spaces, we conclude that (C, p_∞) is an Alexandrov space of curvature ≥ 0. $\square$

3.6 The asymptotic cone

Proposition 3.6.1. *Let (X, d_X) be an Alexandrov space of curvature ≥ 0 and $p \in X$. Then $(\lambda X, p)$ Gromov-Hausdorff converges as $\lambda \to 0$ to a metric cone (C, p_∞) which is also an Alexandrov space of curvature ≥ 0. Moreover, the isometry class of C does not depend on the choice of p.*

If there is an isometric embedding of the Euclidean space $\mathbb{R}^n \hookrightarrow C$ for some n, then there is also an isometric embedding $\mathbb{R}^n \hookrightarrow X$.

We call (C, p_∞) the *asymptotic cone* of X.

Proof. We define a pseudometric on the set

$$N' := \{\sigma : [0, \infty) \to X \ : \ \sigma \text{ ray}, \ \sigma(0) = p\}$$

by

$$d_{N'}(\sigma_1, \sigma_2) := \lim_{s,t\to\infty} \widetilde{\sphericalangle}\sigma_1(s)p\sigma_2(t)$$

Observe that the limit is well defined (by property (B) in the definition of Alexandrov spaces) and that $d_{N'}$ is a pseudometric with $d_{N'} \leq \pi$. Let (N, d_N) be the metric space obtained by quotienting out points whose distance is 0. Since closed distance balls around p are compact and by property (B) in the definition of Alexandrov spaces, we conclude that N is compact. Let (C, p_∞) be the metric cone over N. Then for $(t_1, [\sigma_1]), (t_2, [\sigma_2]) \in C$ we have

$$d_C\left((t_1, [\sigma_1]), (t_2, [\sigma_2])\right) = \lim_{\lambda\to 0} \lambda d_X\left(\sigma_1(\tfrac{t_1}{\lambda}), \sigma_2(\tfrac{t_2}{\lambda})\right) \tag{3.10}$$

and $\lambda d(\sigma_1(\frac{t_1}{\lambda}), \sigma_2(\frac{t_2}{\lambda}))$ is nondecreasing in λ.

Now we want to show using property (A$'$) from subsection 3.1.2, that the pointed metric spaces $(\lambda X, p)$ converge to (C, p_∞) as $\lambda \to 0$. Fix some $\varepsilon > 0$ and $r < \infty$. Since $B_r^C(p_\infty)$ is totally bounded, we may choose an $\frac{\varepsilon}{2}$-net $\{x_0 = p_\infty, x_1, \ldots, x_m\}$ of $B_r^C(p_\infty)$. Write $x_i = (t_i, [\sigma_i])$. For any $\lambda > 0$ set $x_i^\lambda := \sigma_i(\frac{t_i}{\lambda})$.

We first show that $\{x_0^\lambda = p, x_1^\lambda, \ldots, x_m^\lambda\}$ is an ε-net of $B_r^{\lambda X}(p)$ or equivalently a $\frac{\varepsilon}{\lambda}$-net of $B_r^X(p)$ for small λ. Assume that this is not the case. Then there are a sequence $\lambda_k \to 0$ and a sequence $y_k \in B^X_{r/\lambda_k}(p)$ such that $d_X(y_k, x_i^{\lambda_k}) \geq \frac{\varepsilon}{\lambda_k}$ for any $0 \leq i \leq m$. Let γ_k be minimizing geodesics between p and y_k. Since $l_k := d_X(y_k, p) = d_X(y_k, x_0^{\lambda_k}) \geq \frac{\varepsilon}{\lambda_k} \to \infty$, we may assume after passing to a subsequence that the minimizing geodesics γ_k converge to some ray γ with $\gamma(0) = p$ (note that bounded subsets of X are relatively compact). Now observe that $\varepsilon \leq \lambda_k l_k < r$, so after passing to a subsequence we may also assume that $\lambda_k l_k \to t \in [2\varepsilon, r]$. Set $y' := (t, [\gamma]) \in \overline{B}_r^C(p)$ and choose i such that $d_C(y', x_i) \leq \frac{\varepsilon}{2}$. Then for all k

$$\varepsilon \leq d_{\lambda_k X}(y_k, x_i^{\lambda_k}) \leq \lambda_k d_X\left(y_k, \gamma(\tfrac{t}{\lambda_k})\right) + \lambda_k d_X\left(\gamma(\tfrac{t}{\lambda_k}), \sigma_i(\tfrac{t_i}{\lambda_k})\right). \tag{3.11}$$

Since $\lambda_k l_k \to t$, we conclude that for small λ_k we have $\lambda_k d_X(y_k, \gamma(\frac{t}{\lambda_k})) = \lambda_k d_X(\gamma_k(l_k), \gamma(\frac{t}{\lambda_k})) \leq \lambda_k l_k d_X(\gamma_k(1), \gamma(\frac{t}{\lambda_k l_k})) \to 0$. So by (3.10) we conclude that the right hand side of (3.11) becomes $< \varepsilon$ for large k. A contradiction. So $\{x_0^\lambda, \ldots, x_m^\lambda\}$ is indeed an ε-net of $B_r^{\lambda X}(p)$ for small λ.

In addition, equation (3.10) gives us that for small λ we have

$$\left| d(x_i, x_j) - d(x_i^\lambda, x_j^\lambda) \right| < \varepsilon \qquad \text{for all } 0 \leq i, j \leq m.$$

So in fact, we have pointed Gromov-Hausdorff convergence.

The fact that the asymptotic cone is independend of the base point p and the last assertion of the Proposition are clear. $\square$

Chapter 4

No local collapsing

In this chapterr we prove the No Local Collapsing Theorem that will later give us a usefull lower bound on the injectivity radius in a Ricci flow. We will give a brief overview over the arguments developed in [Per1, Sec 7]. For detailed computations and technical issues see [KL, Sec 14ff], [MT, Chp 6], [Top], [Mül]. Note that [Per1, Sec 9] gives an interpretation of these arguments.

4.1 $\mathcal{L}$-geometry

Let $M \times [0,T]$ be a Ricci flow. Choose a basepoint $(x_0,t_0) \in M \times [0,T]$. Depending on t_0 we introduce a new (backward time) parameter $\tau := t_0 - t$ on $M \times [0,T]$. In the following we will most often parameterize geometric quantities by τ rather than t if there is no chance of confusion, so e.g. $S(x,\tau)$ denotes the scalar curvature at x at time $t_0 - \tau$, $\langle\cdot,\cdot\rangle_\tau$ the metric, ∇^τ the Levi-Civita connection etc.

Definition 4.1.1. *Let $\gamma : [\tau_1,\tau_2] \to M$ be a piecewise smooth curve (where $0 \le \tau_1 < \tau_2 \le t_0$). Imagine γ parameterized in space-time and define the* $\mathcal{L}$-length *of γ by*

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau}\left(\left\|\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right\|_\tau^2 + S(\gamma(\tau),\tau)\right)\mathrm{d}\tau.$$

We want to minimize the $\mathcal{L}$ functional. Associate to γ the velocity vector field $X \in \Gamma^\gamma(TM)$ via $X := \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = -\dot\gamma$. At the non-smooth points we denote by X^- resp. X^+ the left resp. right derivatives. Choose a vector field $Y \in \Gamma^\gamma TM$ along γ and a variation $\gamma_\cdot : (-\delta,\delta) \times [\tau_1,\tau_2] \to M$ with $\gamma_0 = \gamma$ and $\frac{\partial}{\partial s}|_{s=0}\gamma_s(\tau) = Y(\tau)$. We can extend the vector fields X and Y along γ to vector fields along $\gamma_\cdot$ by $X(s,\tau) = \frac{\partial}{\partial\tau}\gamma_s(\tau)$ and $Y(s,\tau) = \frac{\partial}{\partial s}\gamma_s(\tau)$. A straight forward calculation gives

Lemma 4.1.2 (first variation). *Assume that γ has breaking points $\tau_0' = \tau_1 < \tau_2' < \ldots < \tau_k' = \tau_2$. Then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}(\gamma_s) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}\left\langle \nabla S - 2\nabla_X X - 4\operatorname{Ric}(X) - \frac{1}{\tau}X, Y\right\rangle \mathrm{d}\tau \\ + 2\sqrt{\tau}\langle X,Y\rangle_\tau\Big|_{\tau_1}^{\tau_2} + \sum_{i=2}^{k-1} 2\sqrt{\tau_i'}\left\langle X^-(\tau_i') - X^+(\tau_i'), Y(\tau_i')\right\rangle_{\tau_i'}\end{aligned}$$

where the geometric quantities under the integral sign are taken at time τ.

So the Euler-Lagrange equations for $\mathcal{L}$ are (we will always consider the geometric quantities at time τ)

$$\nabla_X X - \frac{1}{2}\nabla S + 2\operatorname{Ric}(X) + \frac{1}{2\tau}X = 0. \tag{4.1}$$

Via $X = \frac{\mathrm{d}\gamma}{\mathrm{d}\tau}$ equation (4.1) becomes an ODE of second order which we call the *$\mathcal{L}$-geodesic equation.* If γ satisfies (4.1), we call γ an *$\mathcal{L}$-geodesic.* If γ has the property that $\mathcal{L}(\gamma) \leq \mathcal{L}(\gamma')$ for any $\gamma' : [\tau_1, \tau_2] \to M$ with $\gamma'(\tau_1) = \gamma(\tau_1)$ and $\gamma'(\tau_2) = \gamma(\tau_2)$, we say that γ is *$\mathcal{L}$-minimizing.* The first variation formula implies that any $\mathcal{L}$-minimizing curve is also $\mathcal{L}$-geodesic.

A reparameterization via $\tau = s^2$ yields for $X' := \frac{\mathrm{d}\gamma}{\mathrm{d}s} = 2sX$

$$\nabla_{X'} X' - 2s^2 \nabla S + 4s \operatorname{Ric}(X') = 0. \tag{4.2}$$

This implies that given any time $0 < \tau \leq \tau_0$ and any vector $X(\tau) \in TM$ we can solve (4.1) forwards and backwards in time and obtain a smooth $\mathcal{L}$-geodesic $\gamma : [0, t_0] \to M$. Furthermore, the limit $v := \lim_{\tau \to 0} \sqrt{\tau} X(\tau) \in T_{x_0} M$ exists and for any $v \in T_{x_0} M$ there is exactly one $\mathcal{L}$-geodesic γ_v with this property. We can now define the *$\mathcal{L}$-exponential map*: For any $v \in T_{x_0} M$ and $\tau > 0$ set $\mathcal{L}\exp^\tau_{x_0,t_0}(v) := \gamma_v(\tau)$. Note that in view of (4.2) we have the convergence

$$\mathcal{L}\exp^\tau_{x_0,t_0}\left(\frac{v}{2\sqrt{\tau}}\right) \xrightarrow[\tau \to 0]{} \exp_{x_0,t_0}(v)$$

which is smooth in v. Here $\exp_{x_0,t_0}$ denotes the exponential map in x_0 at time t_0.

Finally, we mention that we can compute the $\mathcal{L}$-length of γ under the reparameterization $\tau = s^2$ by

$$\mathcal{L}(\gamma) = \int_{\sqrt{\tau_1}}^{\sqrt{\tau_2}} \left(\frac{1}{2}\|X'\|^2 + 2s^2 S(\gamma(s^2), s^2)\right) \mathrm{d}s. \tag{4.3}$$

Let

$$D^\tau_{x_0,t_0} := \left\{ v \in T_{x_0} M \ : \ \begin{array}{l} \gamma_v : [0, \tau + \delta] \to M \\ \text{is } \mathcal{L}\text{-minimizing for some } \delta > 0 \end{array} \right\} \tag{4.4}$$

be the *domain* of $\mathcal{L}\exp$. It is easy to see that $D^\tau_{x_0,t_0}$ is open and $D^{\tau_1}_{x_0,t_0} \supset D^{\tau_2}_{x_0,t_0}$ for $\tau_1 < \tau_2$.

Using the results devoloped so far we are able to prove

Lemma 4.1.3. *$\mathcal{L}$-geodesics $\gamma : [\tau_1, \tau_2] \to M$ are locally minimizing (also at time 0), i.e. for any $\tau \in [\tau_1, \tau_2]$ there is a closed interval $I \subset [\tau_1, \tau_2]$ which is a neighborhood of τ such that $\gamma|_I$ is a minimizing $\mathcal{L}$-geodesic.*

Moreover, if the time slices of $M \times [0, T]$ are complete and the curvature on $M \times [0, T]$ is uniformly bounded, then for any two points $x_1, x_2 \in M$ and any $0 \leq \tau_1 < \tau_2 \leq t_0$ there is a minimizing $\mathcal{L}$-geodesic $\gamma : [\tau_1, \tau_2] \to M$ with $\gamma(\tau_1) = x_1$ and $\gamma(\tau_2) = x_2$.

Assume from now on that M is connected, the time slices of $M \times [0, T]$ are complete and the curvature on $M \times [0, T]$ is uniformly bounded. In this case $\mathcal{L}\exp^\tau_{x_0,t_0}$ is surjective for all τ. Furthermore, we can show that $M \setminus \exp^\tau_{x_0,t_0} D^\tau_{x_0,t_0}$ is a set of measure zero.

For any point $x \in M$ and any time $\tau \in [0, t_0]$ we set

$$L(x, \tau) := \inf \left\{ \mathcal{L}(\gamma) \ : \ \begin{array}{ll} \gamma : [0, \tau] \to M & \text{piecewise smooth} \\ \gamma(0) = x_0, \ \gamma(\tau) = x & \end{array} \right\}.$$

Observe that this infimum is realized.

Put $\bar{L}(x, \tau) := 2\sqrt{\tau} L(x, \tau)$ and $l(x, \tau) := \frac{1}{2\sqrt{\tau}} L(x, \tau)$. Note that l is invariant under parabolic rescaling and $\bar{L}(x, \tau) \to \operatorname{dist}_0^2(x_0, x)$ for $\tau \to 0$ by (4.3). Moreover by (4.3) we conclude $l(\gamma_v(\tau), \tau) \to \|v\|^2$ for $\tau \to 0$.

For on any time slice $M(t_0 - \tau)$ and any two local vector fields A, B put

$$\begin{aligned} H_\tau(A, B) := -\nabla^2_{B,B} S + 2\langle R(A, B)B, A\rangle - 4(\nabla_A \operatorname{Ric})(B, B) + 4(\nabla_B \operatorname{Ric})(A, B) \\ + 2(\nabla_T \operatorname{Ric})(B, B) - 2\|\operatorname{Ric}(B)\|^2 - \frac{1}{\tau}\operatorname{Ric}(B, B) \end{aligned}$$

where all geometric quantities on the right hand side are taken at time τ. Observe that $(\nabla_T \operatorname{Ric})(B,B) = -\frac{\partial}{\partial\tau}\operatorname{Ric}(B,B) + 4\|\operatorname{Ric}(B)\|^2$. For a local orthonormal frame (e_i) set

$$H_\tau(A) := \sum_i H_\tau(A,e_i) = -\triangle S + 2\operatorname{Ric}(A,A) - 2\langle\nabla S, A\rangle - 2\frac{\partial}{\partial\tau}S - 2\|\operatorname{Ric}\|^2 - \frac{1}{\tau}S$$
$$= -\frac{\partial}{\partial\tau}S + 2\operatorname{Ric}(A,A) - 2\langle\nabla S, A\rangle - \frac{1}{\tau}S.$$

Let now $0 < \tau \leq t_0$ and consider an $\mathcal{L}$-geodesic $\gamma : [0,\tau] \to M$ ($\gamma = \gamma_v$ for some $v \in T_{x_0}M$) with associated velocity vector field X. We can compute

$$\frac{\mathrm{d}}{\mathrm{d}\tilde\tau}\left(\|X(\tilde\tau)\|^2_{\tilde\tau} + S(\gamma(\tilde\tau),\tilde\tau)\right) = -H_{\tilde\tau}(X(\tilde\tau)) - \frac{1}{\tilde\tau}\left(S(\gamma(\tilde\tau),\tilde\tau) + \|X(\tilde\tau)\|^2_{\tilde\tau}\right).$$

Integrating this equation gives

$$\tau^{3/2}\left(\|X(\tau)\|^2_\tau + S(\gamma(\tau),\tau)\right) = -K_\tau + \frac{1}{2}\mathcal{L}(\gamma) \tag{4.5}$$

where we put

$$K_\tau := \int_0^\tau \tilde\tau^{3/2} H_{\tilde\tau}(X(\tilde\tau))\mathrm{d}\tilde\tau.$$

We can now compute using (4.5) and the first variation formula that in the barrier sense

$$\frac{\partial}{\partial\tau}\bar L(x,\tau) \leq 4\tau S(x,\tau) + \frac{2}{\sqrt{\tau}}K_\tau. \tag{4.6}$$

Equality holds if $v \in D^\tau_{x_0,t_0}$. Furthermore, again by (4.5) we have for $v \in D^\tau_{x_0,t_0}$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tilde\tau}\right|_{\tilde\tau=\tau} l(\gamma(\tilde\tau),\tilde\tau) = -\frac{1}{2\tau^{3/2}}K_\tau. \tag{4.7}$$

Consider a variation $\gamma_\cdot : (-\delta,\delta)\times[0,\tau] \to M$ of an $\mathcal{L}$-geodesic γ and the associated vector fields X and Y along $\gamma_\cdot$ as above. Note that for any vector field along γ we can find a variation $\gamma_\cdot$ such that $(\nabla_Y Y)(\cdot,\tilde\tau) = 0$ for all $\tilde\tau \in [0,\tau]$.

Lemma 4.1.4 (second variation). *If $(\nabla_Y Y)(0,\tau) = 0$ (only at τ), we have*

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0}\mathcal{L}(\gamma_s) = 2\sqrt{\tau}\langle\nabla_Y X, Y\rangle(0,\tau) = \int_0^\tau \sqrt{\tilde\tau}\,\big[\nabla^2_{Y,Y}S - 2\langle R(X,Y)Y,X\rangle$$
$$+2\|\nabla_X Y\|^2 + 2(\nabla_X \operatorname{Ric})(Y,Y) - 4(\nabla_Y\operatorname{Ric})(X,Y)\big]\,\mathrm{d}\tilde\tau$$

where we consider all geometric quantities under the integral sign at time $\tilde\tau$.

Choose first Y along γ such that

$$(\nabla_{\partial_\tau + X}Y)(\tilde\tau) = \frac{1}{2\tilde\tau}Y(\tilde\tau) \tag{4.8}$$

(this is also often expressed by $\nabla_X Y = -\operatorname{Ric}(Y) + \frac{1}{2\tilde\tau}Y$). Note that this implies $\|Y(\tilde\tau)\|^2 = \frac{\tilde\tau}{\tau}\|Y(\tilde\tau)\|^2$. Consider a variation $\gamma_\cdot$ of Y such that $(\nabla_Y Y)(0,\tau) = 0$. Plugging the defining equation for Y into the second variation formula and integrating by parts gives

$$\operatorname{Hess}_L(Y(\tau),Y(\tau)) \leq \frac{\|Y(\tau)\|^2}{\tau} - 2\sqrt{\tau}\operatorname{Ric}(Y(\tau),Y(\tau)) - \int_0^\tau \sqrt{\tilde\tau}H_{\tilde\tau}(X(\tilde\tau),Y(\tilde\tau))\mathrm{d}\tau \tag{4.9}$$

in the barrier sense. Now for an orthonormal frame in T_xM at time τ solve (4.8) for $Y(\tau) = e_i$ and sum (4.9) over all i. We conclude that in the barrier sense

$$\triangle L(x,\tau) \le \frac{n}{\sqrt{\tau}} - 2\sqrt{\tau}S(x,\tau) - \frac{1}{\tau}K_\tau, \tag{4.10}$$

so together with (4.6)

$$\frac{\partial}{\partial\tau}\bar{L} + \triangle\bar{L} \le 2n. \tag{4.11}$$

Now let $v \in D^\tau_{x_0,t_0}$, choose $\sigma : (-\delta,\delta) \to T_{x_0}M$ such that $\mathcal{L}\exp^\tau_{x_0,t_0} \circ\sigma$ is a time $t_0-\tau$ geodesic and set $Y(\tilde\tau) = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\exp^{\tilde\tau}_{x_0,t_0}(v+\sigma(s))$. We call Y an *$\mathcal{L}$-Jacobi field* along $\gamma = \gamma_v$. Using the second variation formula, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tilde\tau=\tau} \|Y(\tilde\tau)\|^2 &= 2\operatorname{Ric}(Y,Y) + 2\langle\nabla_X Y, Y\rangle(\tau)\\ &= 2\operatorname{Ric}(Y,Y) + 2\langle\nabla_Y X, Y\rangle(\tau) = 2\operatorname{Ric}(Y,Y) + \frac{1}{\tau}\operatorname{Hess}(Y(\tau),Y(\tau)).\end{aligned} \tag{4.12}$$

For any $v' \in T_{x_0}M$ let $J_\tau(v')$ be the Jacobian for the map $\mathcal{L}\exp^\tau_{x_0,t_0} : T_{x_0}M \to M(t_0-\tilde\tau)$ where we equip $T_{x_0}M$ with the time $(\tau =)0$ metric. From (4.2) we get

$$\tau^{-n/2}J_\tau(v') \xrightarrow[\tau\to 0]{} 2^n.$$

Using (4.12) and (4.10) it is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\log J_\tau(v) \le \frac{n}{2\tau} - \frac{1}{2\tau^{3/2}}K.$$

So if we define $l_\tau(v) := l(\mathcal{L}\exp^\tau_{x_0,t_0}(v),\tau)$, we get from (4.7)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\tau^{-n/2}e^{-l_\tau(v)}J_\tau(v) \le 0 \qquad \text{for all} \qquad v \in D^\tau_{x_0,t_0}.$$

Since $l_\tau(v) \to \|v\|^2$ for $\tau \to 0$, we conclude

$$\tau^{-n/2}e^{-l_\tau(v)}J_\tau(v) \nearrow 2^n e^{-\|v\|^2} \qquad \text{for} \qquad \tau \searrow 0. \tag{4.13}$$

In particular, the *reduced volume*

$$\tilde V(\tau) := \int_M \tau^{-n/2}e^{-l(\cdot,\tau)}\mathrm{d}\mu_\tau \tag{4.14}$$

is nonincreasing in τ.

4.2 The No Local Collapsing Theorem

We will not be able to derive uniform lower bounds on the evolution of the injectivity radius in a Ricci flow. However, we will observe the following noncollapsedness phenomenon: If there is an upper bound for the curvature on some parabolic neighborhood around a point, then there is a certain lower bound on the injectivity radius at that point. Paraphrasing, we can say that we have a lower bound on the injectivity radius on a local scale.

Note that by Proposition 3.4.1 we can make this behaviour precise in the following

Definition 4.2.1 (κ-noncollapsedness). *Let, $\kappa,\rho > 0$, $M \times I$ be a Ricci flow on an n dimensional manifold M and $(x_0,t_0) \in M \times I$. We say that $M \times I$ is* κ-noncollapsed on scales $< \rho$ *at (x_0,t_0) if for all $0 < r < \rho$ for which*

(i) *the ball* $B_r(x_0,t_0)$ *is relatively compact in* M
(ii) *the interval* $[t_0-r^2,t_0]$ *is contained in* I *and*
(iii) $\|R\|<\frac{1}{r^2}$ *on* $P(x_0,t_0,r,-r^2)$
we have $\operatorname{vol}_{t_0} B_r(x_0,t_0)\geq \kappa r^n$. *In the case* $\rho=\infty$ *we say that* $M\times I$ *is* κ-noncollapsed (on all scales) *at* x_0.

Observe that the κ-noncollapsedness on all scales property is invariant under parabolic rescaling. On manifolds we define

Definition 4.2.2. *Let,* $\kappa,\rho>0$, (M,g) *be an* n *dimensional Riemannian manifold and* $x_0\in M$. *We say that* M *is* κ-noncollapsed on scales $<\rho$ *at* x_0 *if for all* $0<r<\rho$ *for which*
(i) *the ball* $B_r(x_0)$ *is relatively compact in* M
(ii) $\|R\|<\frac{1}{r^2}$ *on* $B_r(x_0)$
we have $\operatorname{vol} B_r(x_0)\geq \kappa r^n$. *In the case* $\rho=\infty$ *we say that* M *is* κ-noncollapsed (on all scales) *at* x_0.

We first show that a lower bound on the reduced volume implies a lower bound on the noncollapsedness:

Lemma 4.2.3. *For any* $V>0$ *and* $n\in\mathbb{N}$ *there is a* $\kappa_n(V)>0$ *such that the following holds:*
Let $M\times[0,T]$ *be a Ricci flow on an* n *dimensional manifold* M *with complete time slices and bounded curvature and* $(x_0,t_0)\in M\times[0,T]$ *a basepoint. Assume that for* $r>0$ *we have* $r^2\leq t_0$ *and* $\|R\|\leq\frac{1}{r^2}$ *on* $P(x_0,t_0,r,-r^2)$. *Then if* $\tilde V(-r^2)>V$ *we have* $\operatorname{vol}_{t_0} B_r(x_0,t_0)\geq\kappa$.

Proof. We follow the lines of [KL, Sec 25]. There is a universal constant $C_1>1$ such that for all $0\leq\tau\leq r^2$ the time $t_0-\tau$ metric on $B_r(x_0,t_0)$ is $e^{C_1\tau}$-bilipschitz to the time t_0 metric. By Shi's estimates we also find a universal constant C_2 such that $\|\nabla S\|\leq\frac{C_2}{r^3}$ on $P(x_0,t_0,\frac12 r,-\frac14 r^2)$.

Choose $0<\alpha\leq 1$ such that

$$2^n\int_{\mathbb{R}^n\setminus B_{\frac{1}{10\alpha}}(0)} e^{-\|v\|^2}\mathrm{d}v\leq\frac{V}{2}$$

and

$$e^{C_1\alpha^2}\left(\frac{1}{5\alpha}+\frac{\alpha C_2}{\sqrt{n-1}}\right)\exp\left(2\alpha^2\sqrt{n-1}\right)\leq\frac{1}{2\alpha}.$$

Observe that both inequalities become true for $\alpha\to 0$.

We first show that for all $v\in T_{x_0}M$ with $\|v\|_0<\frac{1}{10\alpha}$ the $\mathcal{L}$-geodesic γ_v stays in $B_{r/2}(x_0,t_0)$ for times $[0,\alpha^2r^2]$ and that $l(\gamma_v(\tau),\tau)\geq-\alpha\sqrt{n(n-1)}$ for all $0\leq\tau\leq\alpha^2r^2$. Let $\gamma:=\gamma_v|_{[0,\alpha^2r^2]}$ with $\|v\|_0<\frac{1}{10\alpha}$. Reparameterize γ via $\tau=s^2$ and let $X'(s):=\frac{\mathrm{d}}{\mathrm{d}s}\gamma(s^2)$ be the corresponding velocity vector field. Observe that $X'(0)=2v$. (4.2) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\|X'(s)\|_{s^2}^2&=4s^2\langle\nabla S,X'(s)\rangle_{s^2}-8s\operatorname{Ric}_{s^2}(X'(s),X'(s))+4s\operatorname{Ric}_{s^2}(X'(s),X'(s))\\&\leq 4s^2\frac{C_2}{r^3}\|X'(s)\|_{s^2}+4s\frac{\sqrt{n-1}}{r^2}\|X'(s)\|_{s^2}^2\end{aligned}$$

as long as $\gamma(s^2)\in B_{r/2}(x_0,t_0)$. So together with $s\leq\alpha r$

$$\frac{\mathrm{d}}{\mathrm{d}s}\|X'(s)\|_{s^2}\leq 2\alpha^2\frac{C_2}{r}+2\alpha\frac{\sqrt{n-1}}{r}\|X'(s)\|_{s^2}.$$

Using the Gronwall Lemma we get

$$\|X'(s)\|_{s^2} \le \left(2\|v\|_0 + \frac{\alpha C_2}{\sqrt{n-1}}\right) \exp\left(2\alpha^2\sqrt{n-1}\right) - \frac{\alpha C_2}{\sqrt{n-1}}$$

and thus

$$\|X'(s)\|_0 \le e^{C_1 s^2}\left(\frac{1}{5\alpha} + \frac{\alpha C_2}{\sqrt{n-1}}\right) \exp\left(2\alpha^2\sqrt{n-1}\right) \le \frac{1}{2\alpha}.$$

This gives us

$$\operatorname{dist}_0(\gamma(s^2), x_0) \le \int_0^{\alpha r} \|X'(s)\| \mathrm{d}s \le \frac{1}{2} r.$$

So $\gamma \subset B_{r/2}(x_0, t_0)$. Moreover, for $0 < \tau \le \alpha^2 r^2$

$$\begin{aligned} l(\gamma(\tau), \tau) &= \frac{1}{2\sqrt{\tau}} \int_0^{\sqrt{\tau}} \left(\frac{1}{2}\|X'(s)\|^2 + 2s^2 S(\gamma(s^2), s^2)\right) \mathrm{d}s \\ &\ge -\frac{1}{2\sqrt{\tau}} \int_0^{\sqrt{\tau}} 2s^2 \frac{\sqrt{n(n-1)}}{r^2} \mathrm{d}s \ge -\alpha^2\sqrt{n(n-1)} =: -F \end{aligned}$$

Now observe that for $\tau = \alpha^2 r^2$

$$\begin{aligned} V \le \tilde{V}(r^2) \le \tilde{V}(\tau) &= \int_{D^\tau_{x_0,t_0}} \tau^{-n/2} e^{-l_\tau(v)} J_\tau(v) \mathrm{d}v \\ &= \int_{D^\tau_{x_0,t_0} \setminus B_{\frac{1}{10\alpha}}(0)} \tau^{-n/2} e^{-l_\tau(v)} J_\tau(v) \mathrm{d}v + \int_{D^\tau_{x_0,t_0} \cap B_{\frac{1}{10\alpha}}(0)} \tau^{-n/2} e^{-l_\tau(v)} J_\tau(v) \mathrm{d}v \end{aligned}$$

Using (4.13) we conclude

$$V \le \int_{T_{x_0}M \setminus B_{\frac{1}{10\alpha}}(0)} 2^n e^{-\|v\|^2} \mathrm{d}v + \tau^{-n/2} e^F \operatorname{vol}_\tau B_{r/2}(x_0, t_0)$$

and thus

$$\operatorname{vol}_0 B_{r/2}(x_0, t_0) \ge e^{-\frac{n}{2}C_1\alpha^2} \operatorname{vol}_\tau B_{r/2}(x_0, t_0) \ge \frac{V}{2}\alpha^n e^{-F-\frac{n}{2}C_1\alpha^2} r^n =: \kappa(V) r^n. \qquad \square$$

We can now prove our final result

Theorem 4.2.4 (No Local Collapsing Theorem). *For any $\kappa_1, \rho_1, t' > 0$, $T, K < \infty$ and $n \in \mathbb{N}$ there is a $\kappa_2(\kappa_1, \rho_1, t', T, K, n) > 0$ such that the following holds:*
Let $M \times [0, T]$ be a Ricci flow on an n dimensional manifold with complete time slices and uniformly bounded curvature. Assume that
(i) $M(0)$ is κ_1-noncollapsed on scales $< \rho_1$ and
(ii) $\|R\| < K$ on $M \times [0, t']$
Then $M \times [0, T]$ is κ_2-noncollapsed everywhere.

Proof. Let $(x_0, t_0) \in M \times [0, T]$ and $r > 0$ such that $t_0 \ge r^2$ and $\|R\| < \frac{1}{r^2}$ on $P(x_0, t_0, r, -r^2)$. We want to find a universal constant $\kappa_2 > 0$ such that $\operatorname{vol}_{t_0} B_r(x_0, t_0) \ge \kappa_2 r^n$. If $t_0 \le t'$ we can easily find κ_2 since we have a universal bound for the distortion of the metric on $M \times [0, t']$. So assume now that $t_0 > t'$. Consider the reduced volume function $\tilde{V}(\tau)$ with respect to (x_0, t_0). In view of Lemma 4.2.3 and the fact that $\tilde{V}(\tau)$ is nonincreasing in τ, it suffices to give a universal lower bound on $\tilde{V}(t_0)$.

Set $\tau' := t_0 - t'$. Assume that we have $l(\cdot, \tau') \geq \frac{n}{2} + \varepsilon$ on M for some $\varepsilon > 0$. This implies $\bar{L}(\cdot, \tau') \geq 2n\tau' + 4\varepsilon\tau'$. Applying the weak maximum principle[1] to (4.11) we find $\bar{L}(\cdot, \tau') \geq 2n\tau + 4\varepsilon\tau'$ for all $0 < \tau \leq \tau'$ and thus

$$l(\cdot, \tau') \geq 2n + \frac{\varepsilon\tau'}{\tau} \to \infty \qquad \text{for} \qquad \tau \to 0,$$

a contradiction. So there is a point $x \in M$ with $l(x, \tau') \leq \frac{n}{2}$ and thus $L(x, \tau') \leq n\sqrt{t_0 - t'}$.

Consider the ball $B := B_{r_1}(x, 0)$ with $r_1 := \min\{K^{-1/2}, \frac{1}{2}\rho_1\}$. Assumptions (i) and (ii) imply $\mathrm{vol}_0\, B \geq \kappa_1 r_1^n$. Moreover, by assumption (ii) there is a bound $C < \infty$ depending only on t', K, κ_1 and r_1 such that $\mathcal{L}(\sigma) < C$ for all minimizing time 0 geodesics $\sigma : [\tau', t_0] \to M$ between x and a point $y \in B$. Concatenation of a minimizing $\mathcal{L}$-geodesic between (x_0, t_0) and (x, t') with these curves σ yields $L(\cdot, t_0) \leq n\sqrt{t_0 - t'} + C$ on B and thus

$$l(\cdot, t_0) \leq \frac{1}{2\sqrt{t'}} \left(n\sqrt{t_0 - t'} + C \right) =: F \qquad \text{on} \qquad B.$$

So

$$\tilde{V}(t_0) = \int_M t_0^{-n/2} e^{-l(\cdot,\tau)} \mathrm{d}\mu_{t_0} \geq \int_B T^{-n/2} e^{-F} \mathrm{d}\mu_{t_0} \geq T^{-n/2} e^{-F} \kappa_1 r_1^n. \qquad \square$$

[1] We have to be careful since the manifold M need not be compact. Observe that since $L(x, \tau) \to \infty$ for $\mathrm{dist}_0(x, x_0) \to \infty$, we can apply the weak maximum principle for manifolds with boundary.

Chapter 5

κ-solutions

In order to understand the geometry of singularities in a Ricci flow we have to analyze certain model solutions that arise in the limit when we look closer and closer at a singularity and normalize curvature by parabolic rescaling. For example, we have the following theorem which we will refine in chapter 6.

Theorem 5.0.1. *Let $M \times [0, T)$ ($T < \infty$) be a Ricci flow on a compact 3 dimensional manifold defined on a maximal time interval $[0, T)$. Then there is a sequence of times $t_k \nearrow T$ and points $x_k \in M$ such that for $\lambda_k := S^{1/2}(x_k, t_k)$ the sequence of pointed parabolically rescaled Ricci flows $(\lambda_k(M \times [0, t_k]), (x_k, t_k))$ converges to a Ricci flow $(M_\infty \times (-\infty, 0], (x_\infty, 0))$ that has the following properties:*
(a) The metric on every time slice is complete and has nonnegative sectional curvature.
(b) The scalar curvature at time 0 is positive.
(c) The scalar curvature is everywhere bounded from above by 1.
(d) $M \times (-\infty, T]$ is κ-noncollapsed on all scales for some $\kappa > 0$.

Proof. Choose $\varphi > 0$ so small that $M \times [0, T)$ satisifies the property of the Hamilton-Ivey pinching (see Theorem 2.9.1) at time 0. Then this property holds everywhere on $M \times [0, T)$. We can now make conclusions of the following type: If we have an upper bound S_0 on the scalar curvature at some point $(x, t) \in M \times [0, T)$, then there is a lower bound $-X_0$ for the sectional curvature at that point which only depends on S_0 in such a way that $\frac{X_0}{S_0} \to 0$ for $S_0 \to \infty$. So there is also an upper bound Y_0 on the sectional curvature at (x, t) with the property that $\frac{Y_0}{S_0} \to 1$ for $S_0 \to \infty$.

Since $T < \infty$, the scalar curvature on $M \times [0, T)$ must be unbounded. Using the result from the preceding paragraph, this implies that also the scalar curvature is unbounded. We find a sequence $t_k \nearrow T$ such that $\max_M S(\cdot, t_k) \to \infty$ and

$$\max_{M \times [0, t_k]} S = \max_M S(\cdot, t_k).$$

Let $x_k \in M$ be the sequence of points where the maxima are attained at time t_k. Again by the preceding paragraph we get that for large k the sectional curvature on $M \times [0, t_k]$ is bounded from above by $2S(x_k, t_k)$ for large k and from below by $-\delta_k S(x_k, t_k)$ for $\delta_k \to 0$.

By the No Local Collapsing Theorem 4.2.4 we can find a $\kappa > 0$ such that $M \times [0, T)$ is κ-noncollapsed. Consider the parabolically rescaled solutions $M_k \times [0, \lambda_k^2 t_k] := \lambda_k(M \times [0, t_k])$ where $\lambda_k := S^{1/2}(x_k, t_k)$. Obviously, $S \leq 1$ on $M_k \times [0, \lambda_k^2 t_k]$ and the sectional curvature is bounded from below by $-\delta_k$ and from above by 2 for large k. Moreover, $M_k \times [0, \lambda_k^2 t_k]$ is also κ-noncollapsed. Since we have a uniform bound for the curvature, we can use Proposition 3.4.1 to obtain a uniform lower bound on the injectivty radii. Thus the sequence of pointed Ricci flows $(M_k \times [0, \lambda_k^2 t_k], (x_k, t_k))$ subconverges to Ricci flow $(M_\infty \times (-\infty, 0], (x_\infty, 0))$ that satisfies properties (a), (c), (d) and $S(x_\infty, 0) = 1$.

For property (b) note that the scalar curvature is nonnegative on $(M_\infty \times (-\infty, 0], (x_\infty, 0))$. So if there was a point $y \in M_\infty$ with $S(y, 0) = 0$, we could conclude by the strong maximum principle that $S \equiv 0$ on $M_\infty \times (-\infty, 0)$ hence $S(x, 0) = 0$ by continuity. □

We will express the properties of the model solutions $M_\infty \times (-\infty, 0]$ in the following

Definition 5.0.2 (κ-solution). *Let $\kappa > 0$. A Ricci flow defined on $M \times (-\infty, T]$ (for a connected manifold M) is called a* κ-solution *if*
(a) The metric on every time slice is complete and has nonnegative curvature operator.
(b) The scalar curvature at time 0 is positive.
(c) For any compact time interval $I \subset (-\infty, T]$ the Riemannian curvature R is bounded on $M \times I$.
(d) $M \times (-\infty, T]$ is κ-noncollapsed on all scales.

Obviously there are no 1 dimensional κ-solutions.

For simplicity we will assume in the following that $T = 0$. By the Harnack inequality for the Ricci flow (see Theorem 2.8.2) we can estimate the scalar curvatures at two points $(x_1, t_1), (x_2, t_2) \in M \times (-\infty, 0]$ with $t_1 < t_2$ against each other by

$$S(x_2, t_2) \geq S(x_1, t_1) \exp\left(-\frac{\operatorname{dist}^2_{t_1}(x_1, x_2)}{2(t_2 - t_1)} \right).$$

In particular, the scalar curvature is nondecreasing pointwise.

In the following it will be helpful to deal with a slightly more general type of solution that will prove to be equivalent to the notion of a κ-solution in dimensions 2 and 3. We fix some $\kappa > 0$ for the rest of this chapter.

$(*)$ *Suppose $M \times (-\infty, 0]$ is a Ricci flow on a connected manifold M^n such that*
(a) The metric on every time slice is complete and has nonnegative curvature operator.
(b) The scalar curvature at time 0 is positive.
(c) At every point the scalar curvature is nondecreasing in time.
(d) $M \times (-\infty, 0]$ is κ-noncollapsed on all scales.

Observe that if we have a bound on the scalar curvature on $M(0)$, then $M \times (-\infty, 0]$ is already a κ-solution.

Obviously, if $M \times (-\infty, 0]$ satisfies $(*)$, so do the parabolic rescalings $\lambda(M \times (-\infty, 0])$ of $M \times (-\infty, 0]$ (for the same κ). The analogous result holds for κ-solutions.

Another useful property of the noncollapsedness assumption is that we can easily form limits: Let $M_k \times (-\infty, 0]$ be a sequence of solutions satisfying $(*)$ and let $x_k \in M_k$. Assume that there is a function $C : [0, \infty) \to \mathbb{R}$ such that for every A we can bound the scalar curvature $S(\cdot, 0)$ of the time 0 slice on $B_A^{M_k}(x_k, 0)$ by $C(A)$ for large k. Since the scalar curvature is pointwise nondecreasing and the sectional curvatures are nonnegative on the $M_k \times (-\infty, 0]$, we can uniformly bound the Riemannian curvature on $B_A^{M_k}(x_k, 0) \times (-\infty, 0]$ for any A. The κ-noncollapsedness assumption gives us a tool to uniformly bound the injectivity radii at x_k. So by the results of section 3.3 there is a subsequence of the $(M_k \times (-\infty, 0], (x_k, 0))$ that converges to some Ricci flow $(M_\infty \times (-\infty, 0], (x_\infty, 0))$ satisfying (a), (c) and (d) of $(*)$. Now by the strong maximum principle there are two possibilities: Either $S \equiv 0$ on $M_\infty \times (-\infty, 0]$ or $S(\cdot, 0) > 0$ on M. So, for example, if $\liminf_{k\to\infty} S(x_k, 0) > 0$ we know that $M_\infty \times (-\infty, 0]$ satisfies $(*)$. If moreover $C(A) < C'$ for all A, we get that $M_\infty \times (-\infty, 0]$ is even a κ-solution.

Another method that we will use subsequently is the *splitting principle*: Assume that $M \times (-\infty, 0]$ satisfies $(*)$ and assume that $M(0)$ contains a line. By the Ricci splitting Theorem 1.4.3 there is a complete Riemannian manifold of nonnegative Ricci curvature

$(N(0), g)$ such that $M(0) = N(0) \times \mathbb{R}$. From Corollary 2.5.7 we get that $M \times (-\infty, 0]$ is of the form $(N \times \mathbb{R}) \times (-\infty, 0]$ where $N \times (-\infty, 0]$ is a Ricci flow satisfying $(*)$ for $\frac{1}{2}\kappa$ instead of κ. Moreover, if $M \times (-\infty, 0]$ is a κ-solution, then $N \times (-\infty, 0]$ is a $\frac{1}{2}\kappa$-solution.

5.1 The asymptotic geometry

Let $M \times (-\infty, 0]$ satisfy $(*)$. Before we can say anything about the asymptotic geometry of $M \times (-\infty, 0]$ we need to analyze the asymptotic behaviour of the scalar curvature.

Definition 5.1.1 (asymptotic curvature ratio). *Let N be a noncompact complete Riemannian manifold. The* asymptotic curvature *of N is defined by*

$$\mathcal{R}(N) := \limsup_{d(x_0,x)\to\infty} S(x)d^2(x_0, x),$$

where $x_0 \in N$ denotes an arbitrary basepoint. Obviously, $\mathcal{R}(N)$ is independent of the choice of x_0.

Another quantity we wish to analyze is the *asymptotic volume ratio.*

Definition 5.1.2 (asymptotic volume ratio). *If N^n is a complete Riemannian manifold of nonnegative Ricci curvature and $x_0 \in N$ a basepoint, the* asymptotic volume ratio *is defined by*

$$\mathcal{V}(N) = \lim_{r\to\infty} \frac{\operatorname{vol} B_r(x_0)}{r^n}$$

Again, $\mathcal{V}(N)$ is independent of the choice of the basepoint x_0.

Observe that the limit always exists since the quantity $\frac{\operatorname{vol} B_r(x_0)}{r^n}$ is nonincreasing in r by the Bishop-Gromov Theorem 1.4.2. Moreover, $\mathcal{R}$ and $\mathcal{V}$ are scale invariant. The following discussion is based on 11.3 and 11.4 in [Per1] resp. Sec 39ff in [KL].

Proposition 5.1.3. *If $M \times (-\infty, 0]$ satisfies $(*)$ and M is noncompact, we have $\mathcal{R}(M(0)) = \infty$.*

Proof. We may assume that $M \times (-\infty, 0]$ is a κ-solution, because otherwise the hypothesis would be obvious. Since there are no 1-dimensional κ-solutions we have $n \geq 2$.

Choose a basepoint $x_0 \in M$. Since $M(0)$ has nonnegative sectional curvature, Proposition 3.6.1 implies that we have Gromov-Hausdorff convergence

$$(\lambda M(0), x_0) \xrightarrow[\lambda\to 0]{} (C, x_\infty) \tag{5.1}$$

to the asymptotic cone.

Now, assume that $\mathcal{R} < \infty$. Then (C, x_∞) is smooth: For r large enough we may bound the scalar curvature on $M(0) \setminus B_r(x_0)$ by $\frac{\mathcal{R}+1}{r^2}$. So for any r we get for small λ that $S(\cdot, 0) < \frac{1}{\lambda^2}\frac{\mathcal{R}+1}{(r/\lambda)^2} = \frac{\mathcal{R}+1}{r^2}$ on $\lambda M(0) \setminus B_r^{\lambda M}(x_0, 0)$. By the results of section 3.3 (observe that the scalar curvature is pointwise nondecreasing) we conclude that (C, x_∞) is a smooth cone and there is a sequence $\lambda_k \to 0$ for which the convergence in (5.1) is smooth on C_0. Moreover, we can even extend the smooth convergence to the corresponding Ricci flows for a subsequence: There is a Ricci flow on $C_0 \times (-\infty, 0]$ such that the time 0 slice coincides with the metric on C_0 and after passing to a subsequence of (λ_k) we have for the Ricci flows $M_k \times (-\infty, 0] := \lambda_k (M \times (-\infty, 0])$:

$$(M_k \setminus \{x_0\}) \times (-\infty, 0] \xrightarrow[k\to\infty]{} C_0 \times (-\infty, 0]. \tag{5.2}$$

Observe that the curvature operator is nonnegative on $C_0 \times (-\infty, 0]$. Thus we may apply Lemma 2.10.1 and get that the cone C is locally flat away from its tip x_∞ (so actually

$\mathcal{R} = 0$). Moreover, the Riemannian metric on the limiting Ricci flow $C_0 \times (-\infty, 0]$ is everywhere flat and the Ricci flow itself is stationary.

We want to show that the pointed manifolds $(M_k(-1), x_0)$ Gromov-Hausdorff converge to (C, x_∞) in the same sense as in (5.1). It suffices to check that for every r we have $\| \operatorname{dist}_0 - \operatorname{dist}_{-1} \|_\infty \to 0$ on $B_r^{M_k}(x_0, 0)$ for $k \to \infty$. By a compactness argument we find that we only have to prove the following fact: If $y_k, z_k \in M_k$ are sequences such that for $y_\infty, z_\infty \in C$ we have $y_k \to y_\infty$ and $z_k \to z_\infty$ in (5.1), then $\operatorname{dist}_{-1}(y_k, z_k) - \operatorname{dist}_0(y_k, z_k) \to 0$.

Observe that $\operatorname{dist}_0(y_k, z_k) \to \operatorname{dist}_C(y_\infty, z_\infty)$ and $\operatorname{dist}_{-1}(y_k, z_k) \geq \operatorname{dist}_0(y_k, z_k)$. Let γ_∞ be a minimizing geodesic connecting y_∞ and z_∞ in C. If γ_∞ does not hit x_∞, we get from (5.2) and the fact that $C_0 \times (-\infty, 0]$ is stationary that the length of curves $\gamma_k \subset M_k$ approximating γ_∞ are arbitrarily little distorted under the Ricci flow. So $\operatorname{dist}_{-1}(y_k, z_k) - \operatorname{dist}_0(y_k, z_k) \to 0$. Now consider the case in which γ_∞ hits x_∞. We may assume that $y_\infty = x_\infty$ (otherwise we cut γ_∞ into two pieces and use the triangle inequality). Assume first that $z_\infty \neq x_\infty$. Let $\delta > 0$ be small and consider a point y'_∞ on γ_∞ whose distance to x_∞ is less than δ. Choose a sequence of points $y'_k \in M_k$ such that $y'_k \to y'_\infty$ in (5.1). Then $\operatorname{dist}_0(x_0, y'_k) < \delta$ for large k. Moreover by the preceding conclusion we find $\operatorname{dist}_{-1}(y'_k, z_k) - \operatorname{dist}_0(y'_k, z_k) \to 0$. In the following paragraphs we will show that for any $\delta > 0$ we have $B_\delta^{M_k}(x_0, 0) \subset B_{20\delta}^{M_k}(x_0, -1)$ for large k. Using this result and the triangle inequality, we get

$$\begin{aligned} \operatorname{dist}_{-1}(y_k, z_k) - \operatorname{dist}_0(y_k, z_k) &\leq \operatorname{dist}_{-1}(y_k, y'_k) + \operatorname{dist}_{-1}(y'_k, z_k) \\ &\quad - \operatorname{dist}_0(y_k, y'_k) - \operatorname{dist}_{-1}(y'_k, z_k) < 40\delta + \operatorname{dist}_{-1}(y'_k, z_k) - \operatorname{dist}_{-1}(y'_k, z_k) \end{aligned}$$

for large k. In the case $z_\infty = x_\infty$ we immediately get $\operatorname{dist}_{-1}(y_k, z_k) - \operatorname{dist}_0(y_k, z_k) < 40\delta$ for large k. Now choose δ smaller and smaller. It follows that $\operatorname{dist}_{-1}(y_k, z_k) - \operatorname{dist}_0(y_k, z_k) \to 0$ and we have Gromov-Hausdorff convergence

$$(\lambda M(-1), x_0) \xrightarrow[\lambda \to 0]{} (C, x_\infty). \tag{5.3}$$

We have to show that for any $\delta > 0$ we have $B_\delta^{M_k}(x_0, 0) \subset B_{20\delta}^{M_k}(x_0, -1)$ for large k. Let $A := A_{0.1\delta, 3\delta}(x_\infty, 0) \subset C$. By (5.2) for any $\varepsilon > 0$ there are, for large enough k, smooth maps

$$\Phi_k : A \to M_k$$

that are ε-isometries for times $[-1, 0]$ and that converge to id_A in (5.1). Let $\Sigma \subset r^{-1}(2\delta) \subset A$ be a connected component of the distance sphere of radius 2δ. Consider the metric space (N, d_N) with the property $C = \operatorname{Cone}(N)$ (see Definition 1.5.1). By Proposition 1.5.2 is a Riemannian manifold and it is easy to see that a connected component of N is isometric to Σ in the Riemannian sense. So $\operatorname{diam} \Sigma \leq 2\pi\delta$ where we consider the path metric of the induced metric on Σ. Thus Σ is a compact hypersurface and using Proposition 1.5.2 we can compute its shape operator W: Note that $\frac{1}{2\delta} r \partial_r$ is a normal vector field. For any $x \in \Sigma$ and any vector $v \in T_x \Sigma$ we have

$$W(v) = \frac{1}{2\delta} (\nabla_v (r \partial_r))_x = \frac{1}{2\delta} v \qquad \Longrightarrow \qquad W = \frac{1}{2\delta} \operatorname{id}_{T\Sigma}.$$

Now consider the images $\Sigma_k := \Phi_k(\Sigma) \subset M_k$. For ε sufficiently small and large k we can assume that on Σ_k the shape operator satisfies $W^{\Sigma_k} \geq \frac{1}{4\delta}$ in $M_k(-1)$. Let $N \in \Gamma^{\Sigma_k} M_k$ be the time -1 unit normal vector field pointing outward and choose a curve $\sigma : [0, 1] \to M_k$ from x_0 to $x \in \Sigma_k$ such that $\sigma|_{[0,1)}$ does not intersect Σ_k and $\langle \sigma'(1), N_{\sigma(1)} \rangle_{-1} > 0$.

Fix some large k for the moment. Consider the universal cover $\widetilde{M}_k$ of M_k and choose a lift $\widetilde{x}_0$ of x_0. Lift the curve σ starting in $\widetilde{x}_0$ to get $\widetilde{\sigma}$ and denote by $\widetilde{\Sigma} \subset \widetilde{M}_k$ the connected component of the preimage of Σ_k that contains $\widetilde{\sigma}(1)$ under the universal covering projection $\pi : \widetilde{M}_k \to M_k$ (observe that $\widetilde{\Sigma}$ is not necessarily the universal cover of Σ_k). Choose a time

-1 minimizing geodesic $\widetilde{\gamma}:[0,l]\to\widetilde{M_k}$ parameterized by arclength from $\widetilde{x}_0$ to $\widetilde{\Sigma}$ that realizes the distance $\mathrm{dist}_{-1}(\widetilde{x}_0,\widetilde{\Sigma})$ and consider its projection $\gamma=\pi\circ\widetilde{\gamma}$. By the first variation formula (see [dCa, Chp 9]) we get that γ is perpendicular to Σ_k in $\gamma(l)$ at time -1, i.e. $\langle\gamma'(l),N_{\gamma(l)}\rangle_{-1}=\pm1$. Assume that $\langle\gamma'(l),N_{\gamma(l)}\rangle_{-1}=-1$. Then there would be a loop $b:S^1\to\widetilde{M_k}$ intersecting $\widetilde{\Sigma}$ exactly once. This would imply that the intersection number of b and $\widetilde{\Sigma}$ was nonzero. A contradiction to the fact that $\widetilde{M_k}$ is simply connected. So $\langle\gamma'(l),N_{\gamma(l)}\rangle_{-1}=1$.

We want to estimate l from above. Choose a time -1 parallel unit vector field $V(s)$ along γ that is perpendicular to γ. Consider a variation $\gamma_\cdot:(-\theta,\theta)\times[0,l]\to M_k$ such that $\frac{\mathrm{d}}{\mathrm{d}u}\big|_{u=0}\gamma_u(s)=sV(s)$ and $\gamma_u(l)\in\Sigma_k$ for all $u\in(-\theta,\theta)$. Denote by $E(u)$ the energy of γ_u in $M_k(-1)$. Since the variation of γ corresponds to a variation of $\widetilde{\gamma}$ along $\widetilde{\Sigma}$ we have (see [dCa, Chp 9])

$$0\le E''(0)=\int_0^l\left[\langle V(s),V(s)\rangle_{-1}-K_{-1}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}s}\wedge tV(s)\right)\right]\mathrm{d}s-\langle S_{-1}(V(l)),V(l)\rangle_{-1}\le l-\frac{1}{4\delta}l^2$$

So $\mathrm{dist}_{-1}(x_0,\Sigma_k)\le l\le4\delta$. We conclude that for small ε and large k the ball $B^{M_k}_{(4+4\pi)\delta}(x_0,-1)$ covers the inner boundary of $\Phi_k(A_{2\delta,3\delta}(x_\infty,0))$ and thus

$$B^{M_k}_{20\delta}(x_0,-1)\cup\Phi_k(A_{2\delta,3\delta}(x_\infty,0))\supset B^{M_k}_{\delta}(x_0,0).$$

Since $\Phi_k(A_{2\delta,3\delta}(x_\infty,0))$ and $B^{M_k}_{\delta}(x_0,0)$ are disjoint, this implies $B^{M_k}_{\delta}(x_0,0)\subset B^{M_k}_{20\delta}(x_0,-1)$.

Now we apply the Harnack inequality for the Ricci flow (see Theorem 2.8.2) to show that C is actually isometric to Euclidean space. Fix $z_\infty\in C_0$ with $r(z_\infty)<1$ and choose $z_k\in M_k$ converging to z_∞ with respect to the Gromov-Hausdorff convergence (5.1) resp. (5.3). Let A be an arbitrary constant. For every $x\in B^{M_k}_A(x_0,-1)$ we have for large enough k

$$S(x,-1)\le\underbrace{S(z_k,0)}_{\to0}\cdot\underbrace{\exp\left(\frac{\mathrm{dist}_{-1}(x,z_k)}{2}\right)}_{\le\exp\left(\frac{A+1}{2}\right)}\longrightarrow0.$$

So after choosing a subsequence the convergence (5.3) is everywhere smooth and the limit is a flat cone hence isometric to Euclidean space.

By Proposition 3.6.1 we get that $M(0)$ itself is isometric to the Euclidean space $\mathbb{R}^n$ (in the metric sense, hence also in the Riemannian sense). This contradicts the fact that $M\times(\infty,0]$ is a κ-solution. □

Let $M\times(-\infty,0]$ be again a solution satisfying $(*)$ and $x_0\in M$ a basepoint. By Proposition 5.1.3 we know that there is a sequence $x_k\in M$ such that $S(x_k,0)\,\mathrm{dist}_0^2(x_k,x_0)\to\infty$. We want to take a smooth limit of parabolically rescaled copies of $M\times(-\infty,0]$ with basepoint $(x_k,0)$ such that the scalar curvature at $(x_k,0)$ is normalized to 1. In order to do this, we need a bound C such that $S(\cdot,0)<CS(x_k,0)$ on balls $B_{d_k/S^{1/2}(x_k,0)}(x_k,0)$ with $d_k\to\infty$. Observe that all geometric quantities are considered relatively to $S^{-1/2}(x_k,0)$ which is called the *scale at* x_k. We say that we need a uniform bound for the scalar curvature on larger and larger balls around x_k *on the scale at* x_k.

Of course, a priorily we cannot assume this bound. However, if some bound is violated at certain points for certain k, we can replace the x_k by these points and decrease the scale on which the curvature is to be bounded along the way. An iteration of this process is called *point-picking* and will be explained in the following

Lemma 5.1.4 (point-picking). *Let M be a complete Riemannian manifold and $f:M\to\mathbb{R}_+$ a continuous scalar function. Let $x\in M$ and $d>0$. Then there is a $y\in B_{2d/\sqrt{f(x)}}(x)$ such that*

(i) $f(y) \geq f(x)$ *and*
(ii) $f < 4f(y)$ *on* $B_{d/\sqrt{f(y)}}(y)$.

Proof. We describe an algorithm on how to find y. Set $y_0 := x$ and $F_0 := f(y_0)$. Successively apply the following step for $i = 0, 1, 2, \ldots$:

> If $f < 4F_i$ on $B_{d/\sqrt{F_i}}(y_i)$, we are done. If not, choose $y_{i+1} \in B_{d/\sqrt{F_i}}(y_i)$ with $f(y_{i+1}) \geq 4F_i$ and set $F_{i+1} := f(y_{i+1})$.

Observe that we have $F_i \geq 4^i F_0$ and

$$\begin{aligned}\operatorname{dist}(x, y_i) &\leq \operatorname{dist}(y_0, y_1) + \operatorname{dist}(y_1, y_2) + \ldots + \operatorname{dist}(y_{i-1}, y_i) \\ &< \frac{d}{\sqrt{F_0}} + \frac{d}{\sqrt{F_1}} + \ldots + \frac{d}{\sqrt{F_{i-1}}} < \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{i-1}}\right)\frac{d}{\sqrt{F_0}} < \frac{2d}{\sqrt{f(x)}}.\end{aligned}$$

Since f is bounded on $B_{2d/\sqrt{f(x)}}(x)$, the process has to terminate after a finite number of steps. □

Set $d_k := \frac{1}{4}\operatorname{dist}_0(x_0, x_k) S^{1/2}(x_k, 0) \to \infty$. Lemma 5.1.4 gives us a sequence $y_k \in B_{2d_k}(y_k, 0)$ such that we have the estimate $S(\cdot, 0) < 4S(y_k, 0)$ on $B_{d_k/S^{1/2}(y_k,0)}(y_k, 0)$. Moreover, by the triangle inequality

$$\operatorname{dist}_0(x_0, y_k) S^{1/2}(y_k, 0) \geq \frac{1}{2}\operatorname{dist}_0(x_0, x_k) S^{1/2}(x_k, 0) \to \infty. \tag{5.4}$$

Set $y_0 := x_0$ and apply the following

Lemma 5.1.5. *Let (M, g) be a complete Riemannian manifold of nonnegative sectional curvature and $y_k \in M$, $(k \geq 0)$ a sequence with $y_k \to \infty$. Then for a subsequence of the $(y_k)_{k>0}$ there is a ray $\sigma : [0, \infty) \to M$ starting in y_0 and a sequence $s_k \to \infty$ such that* $\operatorname{dist}(y_k, \sigma(s_k)) = \operatorname{dist}(y_0, y_k)$ *and* $\widetilde{\sphericalangle} y_0 y_k \sigma(s_k) \to \pi$.
Furthermore, $\sigma|_{[s_k,\infty)}$ doesn't hit $B_{\frac{1}{2}\operatorname{dist}(y_0,y_k)}(y_k)$.

Proof. Choose minimizing geodesics σ_k between y_0 and y_k. Since $\ell(\sigma_k) \to \infty$ we may assume, after passing to a subsequence, that the σ_k converge to a ray $\sigma : [0, \infty) \to M$ starting in y_0. So $\sphericalangle_{y_0}(\sigma_k, \sigma) \to 0$ and thus for large k we get by Toponogov's Theorem 1.3.1 for $l_k := \operatorname{dist}(y_0, y_k)$

$$\operatorname{dist}(y_k, \sigma(l_k)) < \frac{1}{2} l_k. \tag{5.5}$$

So for large k we can find an $s_k > l_k$ such that $\operatorname{dist}(y_k, \sigma(s_k)) = l_k$. Toponogov's Theorem yields

$$\widetilde{\sphericalangle}\sigma(s_k) y_0 y_k \leq \sphericalangle_{y_0}(\sigma_k, \sigma) \to 0.$$

Since the comparison triangle $\triangle \widetilde{y}_0 \widetilde{\sigma(s_k)} \widetilde{y}_k$ is isosceles, this implies $\widetilde{\sphericalangle} y_0 y_k \sigma(s_k) \to \pi$. It is easy to see that from (5.5) we get that $\sigma|_{[s_k,\infty)}$ does not hit $B_{\frac{1}{2}l_k}(y_k)$ for large k. □

Now parabolically rescale the solution $M \times (-\infty, 0]$ by the factor $\lambda_k = S^{1/2}(y_k, 0)$ and call the resulting flow $M_k \times (-\infty, 0]$. On $M_k \times (-\infty, 0]$ we have $S(y_k, 0) = 1$ and

$$S(\cdot, 0) < 4 \qquad \text{on} \qquad B^{M_k}_{d_k}(y_k, 0).$$

So after passing to a subsequence we have convergence of pointed Ricci flows

$$(M_k \times (-\infty, 0], y_k) \xrightarrow[k\to\infty]{} (M_\infty \times (-\infty, 0], y_\infty) \tag{5.6}$$

for a κ-solution $M_\infty \times (-\infty, 0]$ with $S(y_\infty, 0) = 1$.

Connect the points y_k, y_0 resp. $y_k, \sigma(s_k) \in M_k$ by minimizing geodesics γ_k resp. γ'_k : $[0, l_k] \to M_k(0)$ starting in y_k. From (5.4) we find $l_k \to \infty$ and Toponogov's Theorem implies that for any $l > 0$ we have $\widetilde{\sphericalangle}\gamma_k(l) y_k \gamma'_k(l) \to \pi$. As a consequence the geodesics γ_k resp. γ'_k subconverge on the final time slice of (5.6) to rays γ_∞ resp. $\gamma'_\infty \subset M_\infty(0)$ which connect to a whole line. The splitting principle implies that $M_\infty \times (-\infty, 0] = (N \times \mathbb{R}) \times (-\infty, 0]$ where $N \times (-\infty, 0]$ is an $n-1$ dimensional $\frac{1}{2}\kappa$-solution.

We have proved the following statement on the asymptotic geometry of $M \times (-\infty, 0]$:

Proposition 5.1.6. *If $M \times (-\infty, 0]$ satisfies $(*)$ and M is noncompact, then there is a sequence of points $y_k \in M$ and a sequence of numbers $\lambda_k = S^{1/2}(y_k, 0) > 0$ such that the sequence of parabolically rescaled solutions $(\lambda_k(M \times (-\infty, 0]), (y_k, 0))$ converges to a κ-solution of the form $((N \times \mathbb{R}) \times (-\infty, 0], (y_\infty, 0))$ where $N \times (-\infty, 0]$ is an $n-1$ dimensional $\frac{1}{2}\kappa$-solution.*

This Proposition has some important consequences.

Corollary 5.1.7. *Every 2-dimensional solution $M \times (-\infty, 0]$ satisfying $(*)$ is compact.*

Proof. Otherwise Proposition 5.1.6 would give us a 1-dimensional κ-solution. But there are no 1 dimensional κ-solutions. $\square$

Corollary 5.1.8. *Let $M \times (-\infty, 0]$ satisfy $(*)$. Then $\mathcal{V}(M(0)) = 0$.*

Proof. We will use induction on the dimension n. If $n = 2$, the manifold M is compact and the hypothesis applies. Now assume that M is noncompact. By Proposition 5.1.6 there is a sequence $y_k \in M$ such that for $\lambda_k = S^{1/2}(y_k, 0)$ the parabolically rescaled pointed solutions $(M_k \times (-\infty, 0], (y_k, 0)) = (\lambda_k(M_{\lambda_k} \times (-\infty, 0]), (y_k, 0))$ converge to a κ-solution $((N \times \mathbb{R}) \times (-\infty, 0], (y_\infty, 0))$ where $N \times (-\infty, 0]$ is an $n-1$ dimensional $\frac{1}{2}\kappa$-solution. By the induction hypothesis, $\mathcal{V}(N(0)) = 0$, so

$$\mathcal{V}(N(0) \times \mathbb{R}) = \lim_{r \to \infty} \frac{\operatorname{vol} B_r^{N(0) \times \mathbb{R}}(y_\infty)}{r^n} \leq \lim_{r \to \infty} \frac{\operatorname{vol} B_r^{N(0)}(y_\infty) \cdot 2r}{r^n} = \mathcal{V}(N) = 0.$$

There is a sequence $r_k \to \infty$ such that for any $\varepsilon > 0$ there are ε-isometries

$$\Phi_k : \left(B_{2r_k}^{N(0) \times \mathbb{R}}(y_\infty), y_\infty\right) \longrightarrow (M_k(0), y_k)$$

for large k. So for ε small enough we may conclude from the Bishop-Gromov Theorem 1.4.2

$$\mathcal{V}(M(0)) = \mathcal{V}(M_k(0)) \leq \frac{\operatorname{vol} B_{r_k}^{M_k(0)}(y_k)}{r_k^n} \leq \frac{2 \operatorname{vol} B_{2r_k}^{N(0) \times \mathbb{R}}(y_k)}{r_k^n} \longrightarrow 0. \qquad \square$$

Corollary 5.1.9. *Let $M \times (-\infty, 0]$ be a 2 or 3 dimensional solution satisfying $(*)$. Then $M(0)$ has bounded curvature, hence $M \times (-\infty, 0]$ is already a κ-solution.*

Proof. We just have to show that the scalar curvature is bounded on $M(0)$. In view of Corollary 5.1.7 the case $n = 2$ is trivial. So let $n = 3$. After passing to the universal cover, we may assume that M is simply connected.

Assume that $M(0)$ does not have bounded curvature. Then there is a sequence $x_k \in M$ with $S(x_k, 0) \to \infty$. Let $x_0 \in M$ be a basepoint. Obviously $\operatorname{dist}_0(x_0, x_k) \to \infty$, so $S(x_k, 0) \operatorname{dist}_0^2(x_0, x_k) \to \infty$. Apply the preceding discussion for the points x_k to get points $y_k \in M$, a ray $\sigma : [0, \infty) \to M(0)$, a sequence $s_k \to \infty$ and minimizing geodesics γ_k and γ'_k. We have $S(y_k, 0) \geq S(x_k, 0) \to \infty$ and for the by $\lambda_k := S^{1/2}(y_k, 0) \to \infty$ rescaled pointed solutions $(M_k \times (-\infty, 0], (y_k, 0))$ we proved smooth Gromov-Hausdorff convergence

$$(M_k(0), y_k) \xrightarrow[k \to \infty]{} (N(0) \times \mathbb{R}, y_\infty)$$

where $N(0)$ is the final time slice of a 2 dimensional $\frac{1}{2}\kappa$-solution $N \times (-\infty, 0]$ hence compact. Set $d := \operatorname{diam}_0 N(0)$.

So for any $\varepsilon > 0$ there are ε-isometries

$$\Phi_k : N(0) \times (-2d, 2d) \to \lambda_k M(0) \quad \text{with} \quad y_\infty \mapsto y_k$$

for large k. The sets $U_k := \operatorname{Im} \Phi_k$ are relatively compact. Assume that for some k, the set U_k does not separate $M(0)$. Then we can find a map $b : S^1 \to M$ that intersects $N' := \Phi_k(N \times \{0\}) \subset M$ exactly once and transversely. But this implies that b and N' have nonzero intersection numer contradicting the fact that $M(0)$ is simply connected. So U_k separates M for every k, i.e. $M \setminus U_k$ has two components.

It is easy to see that for large k the points y_0 and $\sigma(s_k)$, being endpoints of an almost straight broken line, lie in different components of $M \setminus U_k$. Since the ray $\sigma|_{[s_k,\infty)}$ does not hit $U_k \subset B_{\frac{1}{2}\operatorname{dist}_0(y_0,y_k)}(y_k)$ (for large k), the component in which $\sigma(s_k)$ lies has to be noncompact for large k. If the other part was noncompact as well, $M(0)$ would have two ends, $M \times (-\infty, 0]$ would geometrically split as $(N \times \mathbb{R}) \times (-\infty, 0]$ and N would be compact. This would contradict the assumption of unbounded curvature.

Now call L_k the noncompact part of $M(0) \setminus U_k$. Since $M \setminus L_k$ is compact and the scalar curvatures on U_j go to ∞ for $j \to \infty$, there can only be a finite number of j's with the property that $U_j \not\subset L_k$. So by passing to a subsequence we may assume that

$$M \supset L_1 \supset L_2 \supset \ldots.$$

and $y_0 \in M \setminus L_1$. Choose $y_0' \in M(0) \setminus L_1$ near y_0 such that any minimizing geodesic between y_0 and y_0' encloses an angle > 0 and $< \pi$ with σ at y_0.

Fix some large k. Let r be so large that $U_k \subset B_r(y_0, 0)$ and $U_k \subset B_r(y_0', 0)$. Choose s' such that for $u := \sigma(s')$ we have $u \in L_k$ and $\operatorname{dist}_0(u, U_k) > r$. Let $\gamma : [0, 1] \to M(0)$ be the geodesic $\sigma|_{[0,s']}$ parameterized by constant speed in the reverse direction (i.e. $\gamma(0) = u$ and $\gamma(1) = y_0$). Furthermore, choose a constant speed minimizing geodesic $\gamma' : [0, 1] \to M(0)$ from u to y_0'. We know that γ and γ' have to pass U_k. Choose $t, t' \in [0, 1]$ such that $z := \gamma(t)$ resp. $z' := \gamma'(t')$ lie in U_k. By the choice of u we know $t, t' > \frac{1}{2}$. Now there are two cases: If $t \le t'$, Toponogov's Theorem 1.3.1 yields for $w := \gamma(\frac{t}{t'})$

$$\operatorname{dist}_0(\gamma, y_0') \le \operatorname{dist}_0(w, y_0') \le \frac{1}{t'} \operatorname{dist}_0(z, z') \le 2 \operatorname{diam}_0 U_k.$$

Analogously if $t \ge t'$, we get for $w' := \gamma'(\frac{t'}{t})$

$$\operatorname{dist}_0(y_0, \gamma') \le \operatorname{dist}_0(y_0, w') \le \frac{1}{t} \operatorname{dist}_0(z, z') \le 2 \operatorname{diam}_0 U_k.$$

From the assumptions on y_0' we have $2 \operatorname{diam}_0 U_k \ge \operatorname{dist}_0(\gamma, y_0') = \operatorname{dist}_0(\sigma, y_0') > 0$ and using the triangle inequality we deduce

$$\begin{aligned} 0 < \operatorname{dist}_0(y_0', y_0) + \operatorname{dist}_0(y_0, \sigma(1)) - \operatorname{dist}_0(y_0', \sigma(1)) \le \\ \operatorname{dist}_0(y_0', y_0) + \operatorname{dist}_0(y_0, u) - \operatorname{dist}_0(y_0', u) \le 2 \operatorname{dist}_0(y_0, \gamma') \le 4 \operatorname{diam}_0 U_k. \end{aligned}$$

But this contradicts the fact that $\operatorname{diam}_0 U_k \to 0$ for $k \to \infty$. □

5.2 Controlling curvature by local collapsedness and vice versa

The fact that the asymptotic volume ratio of any κ-solution is 0 is now useful to show that we can bound the curvature at a point by the local collapsedness.

Lemma 5.2.1 (volume controls curvature). *For any $\alpha > 0$ there is a $C_{\kappa,n}(\alpha) < \infty$ such that the following holds:*
Suppose that $M^n \times (-\infty, 0]$ is a Ricci flow satisfying $(*)$, $(x,t) \in M \times (-\infty, 0]$ and $r > 0$. If $\mathrm{vol}_0 B_r(x,0) > \alpha r^n$ then $S(x,0) < \frac{C}{r^2}$

Proof. Fix some $\alpha, \kappa > 0$ and $n \in \mathbb{N}$. Suppose that there wasn't such C. Then we can find a sequence of n dimensional Ricci flows $M_k \times (-\infty, 0]$ satisfying $(*)$ for κ and a sequence of points $x_k \in M_k$ as well as a sequence r_k such that

$$\mathrm{vol}_0 B(x_k, 0, r_k) > \alpha r_k^n \qquad \text{but} \qquad r_k^2 S(x_k, 0) \to \infty.$$

Now we apply the point-picking Lemma 5.1.4 with $d_k = \frac{1}{2} r_k S^{1/2}(x_k, 0) \to \infty$ in order to get curvature control near x_k. We find that there are points $y_k \in B_{r_k}^{M_k}(x_k)$ such that for $Q_k := S(y_k, 0)$

$$S(\cdot, 0) < 4Q_k \qquad \text{on} \qquad B^{M_k}_{d_k/Q_k^{1/2}}(y_k, 0).$$

The fact that the distance between x_k and y_k is bounded by r_k implies

$$\mathrm{vol}_0 B^{M_k}_{2r_k}(y_k, 0) \geq \alpha r_k^n. \tag{5.7}$$

By parabolic rescaling we can now assume that $Q_k = S(y_k, 0) = 1$, so

$$S(\cdot, 0) < 4 \qquad \text{on} \qquad B^{M_k}_{d_k}(y_k, 0).$$

So after passing to a subsequence we have convergence of pointed Ricci flows

$$(M_k \times (-\infty, 0], (x_k, 0)) \xrightarrow[k \to \infty]{} (M_\infty \times (-\infty, 0], (x_\infty, 0)) \tag{5.8}$$

where $M_\infty \times (-\infty, 0]$ is a κ-solution, hence $\mathcal{V}(M_\infty(0)) = 0$. This implies that we can choose a radius $r > 0$ such that

$$\frac{\mathrm{vol}_0 B_r^{M_\infty}(x_\infty, 0)}{r^n} < \frac{\alpha}{2^n}.$$

From (5.8) one can easily deduce

$$\frac{\mathrm{vol}_0 B_r^{M_k}(y_k, 0)}{r^n} \longrightarrow \frac{\mathrm{vol}_0 B_r^{M_\infty}(y_\infty, 0)}{r^n} < \frac{\alpha}{2^n} \qquad \text{for} \qquad k \longrightarrow \infty$$

On the other hand, we can assume $2r_k > r$ since $r_k \to \infty$ and Bishop-Gromov's Theorem 1.4.2 gives us a contradiction:

$$\frac{\mathrm{vol}_0 B_r^{M_k}(y_k, 0)}{r^n} \geq \frac{\mathrm{vol}_0 B_{2r_k}^{M_k}(y_k, 0)}{(2r_k)^n} \geq \frac{\alpha}{2^n}.$$ □

Corollary 5.2.2. *For any $\alpha > 0$ and $A < \infty$ there is a $C'_{\kappa,n}(\alpha, A) < \infty$ such that the following holds:*
Suppose that $(x, 0)$ is a point on an n dimensional Ricci flow $M \times (-\infty, 0]$ satisfying the assumptions of $(*)$ such that $\mathrm{vol}_0 B_1(x,t) > \alpha$. Then we have $S(\cdot, t) < C'$ on $B_A(x,t)$.

Proof. For any point $y \in B(x, t, A)$ we have

$$\frac{\mathrm{vol}_0 B(y, t, A+1)}{(A+1)^n} \geq \frac{\alpha}{(A+1)^n}$$

So by Lemma 5.2.1

$$S(y, 0) < \frac{C_{\kappa,n}\left(\frac{\alpha}{(A+1)^n}\right)}{(A+1)^2}.$$ □

Corollary 5.2.3. *Let $(M_k \times (-\infty, 0], (x_k, 0))$ be a sequence of n dimensional Ricci flows satisfying the assumptions in $(*)$ for a certain κ such that $\operatorname{vol}_0 B_1(x_k, 0) > \alpha > 0$. Then there is a subsequence $(M_{k_i} \times (-\infty, 0], (x_{k_i}, 0))$ that smoothly converges for $i \to \infty$ to a Ricci flow $(M_\infty \times (-\infty, 0], (x_\infty, 0))$.*

Moreover, $\operatorname{vol}_0 B_1(x_\infty, 0) = \lim_{i\to\infty} \operatorname{vol}_0 B_1^{M_k}(x_{k_i}, 0)$ and $S(x_\infty, 0) = \lim_{i\to\infty} S(x_i, 0)$ and if $S(x_\infty, 0) > 0$ the Ricci flow $M_\infty \times (-\infty, 0]$ satisfies $(*)$.

Proof. Obvious. □

The κ-noncollapsedness gives us a tool to bound the local collapsedness by a local bound on the curvature. We will show that we can even control the local collapsedness by the curvature at just a single point.

Lemma 5.2.4 (curvature controls volume). *There is a $\beta_n > 0$ such that the following holds:*
If (x, t) is a point on a Ricci flow $M^n \times (\infty, 0]$ satisfying $(*)$ and $r = S^{-1/2}(x, 0)$, then

$$\frac{\operatorname{vol}_0 B_r(x, 0)}{r^n} > \beta$$

Proof. Observe that the hypothesis of the Lemma is scale-invariant. Assume that there was no such β_n. Then there is a sequence $(M_k \times (-\infty, 0], (x_k, 0))$ of pointed n dimensional Ricci flows satisfying $(*)$ such that for $r_k := S^{-1/2}(x_k, t)$ we have

$$\frac{\operatorname{vol}_0 B_{r_k}^{M_k}(x_k, 0)}{r_k^n} \longrightarrow 0. \tag{5.9}$$

By Bishop-Gromov's Theorem 1.4.2 we get that $\frac{\operatorname{vol}_0 B(x_k, 0, s)}{s^n} \nearrow \omega_n$ for $s \searrow 0$ where ω_n denotes the volume of the unit ball in n dimensional Euclidean space. So for large k we find $s_k < r_k$ such that

$$\frac{\operatorname{vol}_0 B_{s_k}^{M_k}(x_k, 0)}{s_k^n} = \frac{\omega_n}{2}.$$

By rescaling, we may assume that $s_k = 1$. Since $\operatorname{vol}_0 B_{r_k}^{M_k}(x_k, 0) \geq \operatorname{vol}_0 B_1^{M_k}(x_k, 0) = \frac{\omega_n}{2}$ the convergence (5.9) implies $r_k \to \infty$. By Corollary 5.2.3 we get smooth convergence of pointed Ricci flows after passing to a subsequence

$$(M_k \times (-\infty, 0], (x_k, 0)) \xrightarrow[k\to\infty]{} (M_\infty \times (-\infty, 0], (x_\infty, 0))$$

where $M_\infty \times (-\infty, 0]$ is an ancient κ-noncollapsed Ricci flow with

$$\operatorname{vol}_0 B_1(x_\infty, 0) = \frac{\omega_n}{2} \qquad \text{and} \qquad S(x_\infty, 0) = \lim_{i\to\infty} \frac{1}{r_{k_i}^2} = 0. \tag{5.10}$$

So by the strong maximum principle we may conclude $S \equiv 0$ on $M_\infty \times (-\infty, 0]$, thus $M_\infty(0)$ is flat, and $M_\infty(0) \cong \mathbb{R}^n/\Gamma$ for a discrete subgroup $\Gamma \triangleleft \operatorname{Isom}(\mathbb{R}^n)$ acting freely on $\mathbb{R}^n$. Because of (5.10), $M_\infty(0)$ can not be isometric to $\mathbb{R}^n$, hence $\Gamma \neq 1$. But by Bieberbach's Theorem 1.4.6 there is a Riemannian covering $T^k \times \mathbb{R}^{n-k} \to \mathbb{R}^n/\Gamma$ with $k > 0$ where T^k is a k-dimensional flat torus. Using the κ-noncollapsedness of $M_\infty \times (-\infty, 0]$, we find $\kappa \leq \mathcal{V}(M_\infty) \leq \mathcal{V}(T^k \times \mathbb{R}^{n-k}) = 0$, a contradiction. □

Corollary 5.2.5. *For any $A < \infty$ there is a $C_{\kappa,n}(A) < \infty$ such that the following holds: Let $M \times (-\infty, 0]$ be an n dimensional Ricci flow satisfying $(*)$ and $x \in M$. Then*

$$\frac{1}{C} S(x, 0) < S(\cdot, 0) < C S(x, 0) \qquad \text{on} \qquad B_{AS^{-1/2}(x,0)}(x, 0).$$

Proof. Observe that the hypothesis is scale invariant. Thus we may assume that $S(x,0) = 1$. Applying Lemma 5.2.4 we get $\operatorname{vol}_0 B_1(x,0) > \beta$. So Corollary 5.2.2 gives $S < C'_{\kappa,n}(\beta, C_1)$ on $B_{AS^{-1/2}(x,0)}(x,0)$. We have established the upper bound.

For the lower bound choose $y \in B_{AS^{-1/2}(x,0)}(x,0)$. If $S(y,0) > S(x,0)$, there is nothing to show. If $S(y,0) \le S(x,0)$ we have $x \in B_{AS^{-1/2}(y,0)}(y,0)$ and we can apply the Corollary for y. □

As a special case we formulate the following Corollary which compares the distance between two points on the scale of each point.

Corollary 5.2.6. *For any $C_1 < \infty$ there is a $C_2(C_1, \kappa, n) < \infty$ such that the following holds:*
If $M \times (-\infty, 0]$ is an n dimensional Ricci flow satisfying $(*)$ and $x, y \in M$, we have

$$S(x,0)\operatorname{dist}_0^2(x,y) < C_1 \qquad \Longrightarrow \qquad S(y,0)\operatorname{dist}_0^2(x,y) < C_2$$

Corollary 5.2.7. *Let $(M_k \times (-\infty, 0], (x_k, 0))$ be a sequence of pointed solutions satisfying $(*)$ for a certain $\kappa > 0$ such that $S(x_k, 0) \to \rho_\infty$. Then there is a subsequence $(M_{k_i} \times (-\infty, 0], (x_{k_i}, 0))$ that smoothly converges to either an ancient flat solution if $\rho_\infty = 0$ or if $\rho_\infty > 0$ to an ancient solution $M_\infty \times (-\infty, 0]$ satisfying $(*)$ with $S(x_\infty, 0) = \rho_\infty$.*

In other words: The set of n dimensional pointed solutions satisfying $(*)$ is compact modulo scaling.

Proof. Use Lemma 5.2.4 and Corollary 5.2.3. □

Especially in dimension 2 and 3 this gives (see Corollary 5.1.9):

Proposition 5.2.8 (Compactness of the space of 2 or 3 dimensional pointed κ-solutions modulo scaling). *Let $(M_k \times (-\infty, 0], (x_k, 0))$ be a sequence of pointed 2 or 3 dimensional κ-solutions such that $S(x_k, 0) \to \rho_\infty$. Then there is a subsequence $(M_{k_i} \times (-\infty, 0], (x_{k_i}, 0))$ that smoothly converges to either an ancient flat solution if $\rho_\infty = 0$ or a κ-solution $M_\infty \times (-\infty, 0]$ with $S(x_\infty, 0) = \rho_\infty$ if $\rho_\infty > 0$.*

Finally, we mention a simple application of Corollary 5.2.7. Observe that the quantities in the hypothesis are invariant under parabolic rescaling.

Corollary 5.2.9. *There is a constant $\eta_{\kappa,n} < \infty$ such that for any n dimensional Ricci flow $M \times (-\infty, 0]$ satisfying $(*)$ and any point $x \in M$ we have*

$$\|\nabla S^{-1/2}(x,0)\| < \frac{\eta}{2} \qquad \textit{and} \qquad |\partial_t S^{-1}(x,0)| < \left(\frac{\eta}{2}\right)^2.$$

The factor $\frac{1}{2}$ is chosen in order to simplify future computations.

5.3 Classification of 2 dimensional κ-solutions

From Corollary 5.1.7 we already know that every 2 dimensional solution $M \times (-\infty, 0]$ satisfying $(*)$ is a compact κ-solution and thus is diffeomorphic to either S^2 or $\mathbb{R}P^2$. As for the geometry, Proposition 5.2.8 gives us the following statement.

Lemma 5.3.1. *Let $(M_k \times (-\infty, 0])$ be a sequence of 2 dimensional κ-solutions on S^2 whose volume is normalized, i.e. $\operatorname{vol}_0 M_k = 4\pi$. Then the sequence subconverges to a volume normalized κ-solution $M_\infty \times (-\infty, 0]$ on S^2*

Proof. Choose basepoints $x_k \in M_k$ and consider the sequence pointed rescales solutions $(\lambda_k(M_k \times (-\infty, 0]), (x_k, 0))$ with $\lambda_k = S^{1/2}(x_k, 0)$. On the rescaled solutions $S(x_k, 0) = 1$, so by Proposition 5.2.8 we have convergence to a pointed 2 dimensional κ-solution $(M_\infty \times (-\infty, 0], (x_\infty, 0))$ for a subsequence of indices (k_i). By Corollary 5.1.7 the manifold M_∞ must be compact and thus also diffeomorphic to S^2. For the volume we have

$$\mathrm{vol}_0\, M_{k_i} = 4\pi\lambda_{k_i}^2 \xrightarrow[i\to\infty]{} \mathrm{vol}_0\, M_\infty.$$

Thus the λ_{k_i} converge and by Proposition 5.2.8 again we already have subconvergence of the pointed κ-solutions $(M_k \times (-\infty, 0], (x_k, 0))$ to a pointed κ-solution $(M'_\infty \times (-\infty, 0], (x_\infty, 0))$ (which is indeed equal to $M_\infty \times (-\infty, 0]$). By the same argument, as above M'_∞ is diffeomorphic to S^2 and we have $\mathrm{vol}_0\, M'_\infty = 4\pi$. □

In order deduce some exact geometric statements we need to introduce a scale invariant quantity for Riemannian metrics on S^2 which will turn out to be monotone under the Ricci flow.

Definition 5.3.2. *Let M be a surface of positive (scalar) curvature diffeomorphic to S^2 and denote $V := \mathrm{vol}(M)$ its volume. The* entropy *of M is defined by*

$$N(M) := -\int_M S \log\left(\frac{SV}{8\pi}\right) \mathrm{d}\mu.$$

Lemma 5.3.3. *If M is diffeomorphic to S^2 and has positive curvature, then $N \leq 0$ and $N = 0$ if and only if M is homothetic to the round S^2.*

Proof. Since the function $f : x \mapsto -Vx \log\left(\frac{Vx}{8\pi}\right)$ is concave, we obtain by Jensen's inequality and the Gauß-Bonnet Theorem 1.4.4

$$N(M) = \frac{1}{V}\int_M f(S)\mathrm{d}\mu \leq f\left(\frac{1}{V}\int_M S\mathrm{d}\mu\right) = f\left(\frac{8\pi}{V}\right) = 0.$$

The rest follows easily. □

Recall that for the Ricci flow on a surface the scalar curvature S satisfies the following evolution equation:

$$\dot{S} = \triangle S + S^2.$$

Moreover, if we form the Riemannian measure μ_t at time t, we have $\frac{\mathrm{d}}{\mathrm{d}t}\mu_t = -S\mu_t$.

Lemma 5.3.4. *Let $M \times [0, T]$ be a Ricci flow on a surface M diffeomorphic to S^2 such that the curvature is everywhere positive. Then $\dot{N}(M) \geq 0$. Furthermore, if $\dot{N}_t(M) = 0$ for some $t \in [0, T]$, then $M(t)$ is homothetic to the round S^2.*

Proof. Without loss of generality we assume that $V = 4\pi$. In the following we will always integrate over M and denote the Riemannian measure at time t by $\mathrm{d}\mu$. The Gauß-Bonnet Theorem gives us that $\dot{V} = -8\pi$ and

$$\begin{aligned}\dot{N} &= -\int (\triangle S + S^2)\log\left(\frac{SV}{8\pi}\right)\mathrm{d}\mu - \int (\triangle S + S^2)\,\mathrm{d}\mu + 2\int S\mathrm{d}\mu + \int S^2 \log\left(\frac{SV}{8\pi}\right)\mathrm{d}\mu \\ &= \int \frac{\|\nabla S\|^2}{S}\mathrm{d}\mu - \int S^2\mathrm{d}\mu + 2\int S\mathrm{d}\mu = \int \frac{\|\nabla S\|^2}{S}\mathrm{d}\mu - \int (S-2)^2\mathrm{d}\mu.\end{aligned}$$

The scalar function $S - 2$ has integral 0. So we find an $f \in C^\infty(M)$ with $\triangle f = S - 2$. Set

$$H := \nabla^2 f - \frac{1}{2}\triangle f\langle\cdot,\cdot\rangle = (\nabla^2 f)_0 \ \in \Gamma(\mathrm{Sym}_2 T^*M) \quad \text{and} \quad X := \nabla S + S\nabla f \ \in \Gamma(TM).$$

(Note that $\frac{1}{2}X = \operatorname{div} H = \sum_i \nabla_{E_i} H(E_i, \cdot)$ for an orthonormal frame (E_i). If f is the potential of a gradient shrinking 2-dimensional soliton we have $H \equiv 0$.) Using Stokes' theorem

$$\int \frac{\|X\|^2}{S} \mathrm{d}\mu = \int \frac{\|\nabla S\|^2}{S} \mathrm{d}\mu + \int S \|\nabla f\|^2 \mathrm{d}\mu + \underbrace{2\int \langle \nabla S, \nabla f\rangle \mathrm{d}\mu}_{-2\int S(S-2)\mathrm{d}\mu = -2\int (S-2)^2 \mathrm{d}\mu}$$

This gives us

$$\dot{N} = \int \frac{\|X\|^2}{S} \mathrm{d}\mu - \int S\|\nabla f\|^2 \mathrm{d}\mu + \int (S-2)^2 \mathrm{d}\mu.$$

Observe that we have

$$\|H\|^2 = \|\nabla^2 f\|^2 - \triangle f \left\langle \nabla^2 f, \langle\cdot,\cdot\rangle\right\rangle + \frac{1}{2}(\triangle f)^2 = \|\nabla^2 f\|^2 - \frac{1}{2}(\triangle f)^2 .$$

So using Stokes' theorem again we get

$$\begin{aligned} 2\int \|H\|^2 \mathrm{d}\mu &= 2\int \|\nabla^2 f\|^2 \mathrm{d}\mu - 2\int (\triangle f)^2 \mathrm{d}\mu + \int (S-2)^2 \mathrm{d}\mu \\ &= -2\int \langle \nabla f, \triangle \nabla f\rangle \mathrm{d}\mu + 2\int \langle \nabla f, \nabla \triangle f\rangle \mathrm{d}\mu + \int (S-2)^2 \mathrm{d}\mu \\ &= -\int S\|\nabla f\|^2 \mathrm{d}\mu + \int (S-2)^2 \mathrm{d}\mu \end{aligned}$$

We conclude

$$\dot{N} = \int_M \frac{\|X\|^2}{S} \mathrm{d}\mu + 2\int_M \|H\|^2 \mathrm{d}\mu \geq 0.$$

Now suppose $\dot{N} = 0$ but $S \not\equiv \text{const}$. It follows $H = \nabla^2 f - \frac{1}{2}\triangle f \langle\cdot,\cdot\rangle = \left(\nabla^2 f\right)_0 = 0$. Since

$$\nabla^2 f = \frac{1}{2}\triangle f \langle\cdot,\cdot,\rangle = \operatorname{Ric} - \langle\cdot,\cdot,\rangle,$$

the M is a gradient shrinking soliton. Following [CLT] we will show that there is no gradient shrinking soliton on S^2 other than the round solution.

Assume that $S \not\equiv \text{const}$ and thus $f \not\equiv \text{const}$. Consider the vector field $Y := J\nabla f$ where J denotes the counterclockwise rotation by $\frac{\pi}{2}$ on TM. Since $\nabla . Y = J\nabla . \nabla f = \frac{1}{2}\triangle f J$ is a skew symmetric endomorphism, Y is Killing and M is rotationally symmetric. So f has exactly two critical points $x_{\mathrm{min}\,/\,\mathrm{max}} \in M$ where it assumes its extrema $f_{\mathrm{min}\,/\,\mathrm{max}}$. Let $\gamma : [0, a] \to M$ be a geodesic between x_{min} and x_{max} parameterized by arclength. We can choose polar coordinate (r, θ) on $M \setminus \{x_{\mathrm{min}\,/\,\mathrm{max}}\}$ around x_{min} such that we have for the metric $g = \mathrm{d}r^2 + h^2(r)\mathrm{d}\theta^2$ with $h = u\|\nabla f\|$ for some $u \neq 0$ and γ is the line $\theta \equiv 0$. Then $\nabla f = \frac{\mathrm{d}f}{\mathrm{d}r}\partial_r$. We get

$$h' = u\frac{\langle \nabla_{\partial_r} \nabla f, \nabla f\rangle}{\|\nabla f\|} = u\nabla^2_{\partial_r,\partial_r} f = u\frac{1}{2}S - u = -u\frac{h''}{h} - u.$$

Multiplying by hh' and integrating gives

$$\int_0^a h\left(h'\right)^2 \mathrm{d}r = -\frac{u}{2}\left(h'\right)^2\Big|_0^a - \frac{u}{2}h^2\Big|_0^a .$$

Obviously $h(0) = h(a) = 0$, so the second term on the right hand side vanishes. Furthermore, we must have $h'(0) = -h'(a) = 1$. Thus also the first term vanishes and we conclude $h\left(h'\right)^2 \equiv 0$, but this is impossible. $\square$

Now we are able to prove the final statement

Theorem 5.3.5. *Every 2-dimensional κ-solution is homothetic to the round shrinking $S^2 \times (-\infty, 0]$ or $\mathbb{R}P^2 \times (-\infty]$.*

Proof. Let $\mathcal{S}_{\text{vol}}$ be the set of all normalized (i.e. $\text{vol}_0 = 4\pi$) 2 dimensional κ-solutions diffeomorphic to S^2. It will be enough to show that $\mathcal{S}_{\text{vol}}$ just contains the round shrinking $S^2 \times (-\infty, 0]$. As follows easily from Proposition 5.2.8, $\mathcal{S}_{\text{vol}}$ is compact, i.e. every sequence of Ricci flows of $\mathcal{S}_{\text{vol}}$ subconverges to a Ricci flow in $\mathcal{S}_{\text{vol}}$.

Next observe that the entropy of the final time slice $N_0 : \mathcal{S}_{\text{vol}} \to (-\infty, 0]$ is a continuous function, i.e. for every convergent sequence of Ricci flows $M_k \times (-\infty, 0] \in \mathcal{S}_{\text{vol}}$ the time 0 entropies $N_0(M_k)$ converge to the time 0 entropy if its limit. It is easy to conclude that the image I of N_0 is compact and by Lemma 5.3.3 we have $I \subset (-\infty, 0]$. Let $N_{\min}$ be the minimum of I and $M \times (-\infty, 0]$ be a κ-solution with $N_0(M) = N_{\min}$. By Lemma 5.3.4 the entropy $N_t(M)$ has to be constant in t. So $M \times (-\infty, 0]$ is isometric to the round shrinking S^2 and thus $N_{\min} = 0$. This implies that any $M \times (-\infty, 0] \in \mathcal{S}_{\text{vol}}$ has $N = 0$ and thus is isometric to the round shrinking S^2. ☐

5.4 Classification of 3 dimensional κ-solutions

We will now classify the geometry of 3 dimensional κ-solutions. As explained in section 1.4, every 3 dimensional nonflat complete Riemannian manifold M of nonnegative curvature is isometric to a quotient of $N \times \mathbb{R}$, where N is diffeomorphic to $\mathbb{R}^3$, or M is diffeomorphic to a spherical space form, to $\mathbb{R}^3$, or to one of the following manifolds:

$$S^2 \times \mathbb{R},\ \ \mathbb{R}P^2 \times \mathbb{R},\ \ S^2 \tilde{\times} \mathbb{R},\ \ S^2 \times S^1,\ \ \mathbb{R}P^2 \times S^1,\ \ S^2 \tilde{\times} S^1,\ \ \mathbb{R}P^3 \# \mathbb{R}P^3.$$

(Here $S^2 \tilde{\times} \mathbb{R} = S^2 \times \mathbb{R}/(a, b) \sim (-a, -b)$ and $S^2 \tilde{\times} S^2 = S^2 \times S^1/(a, b) \sim (-a, -b)$.) By the splitting principle and Corollary 5.1.7, the final time slice of a κ-solution cannot be isometric to a quotient of $N \times \mathbb{R}$ with $N \approx \mathbb{R}^2$. Again using the splitting principle and Theorem 5.3.5, it is furthermore easy to conclude that if a κ-solution is diffeomorphic to one of the latter 7 manifolds, then it is actually *homothetic* to the corresponding quotient of the round shrinking cylinder. In the latter 4 cases (the compact ones) this puts us into trouble concerning the noncollapsedness condition:

Lemma 5.4.1. *There is no κ-solution diffeomorphic to a compact metric quotient of the round cylinder $S^2 \times \mathbb{R}$. Hence all compact κ-solutions are spherical space forms.*

Proof. Assume that $M \times (-\infty, 0]$ is a κ-solution on a compact metric quotient of the round cylinder $S^2 \times \mathbb{R}$. Then $M \times (-\infty, 0]$ is homothetic to the corresponding quotient of the standard round shrinking cylinder $(S^2 \times \mathbb{R}) \times (-\infty, 0]$. Assume that it is even *isometric* to this quotient. Consider the covering map $\pi : S^2 \times \mathbb{R} \to M$. The scalar curvature S on M satisfies $S_t = S(\cdot, t) \equiv \frac{1}{1-t}$. Obviously, there is an A such that $\pi(S^2 \times (-A, A)) = M$. So

$$\text{vol}_t M \leq \text{vol}_t S^2 \times (-A, A) = 16A\pi(1-t). \tag{5.11}$$

Set $r_t^2 := 1/\|R_t\| = \frac{\sqrt{2}}{S_t} = \sqrt{2}(1-t)$. Then for any point $x \in M$ we have $\text{vol}_t M \geq \text{vol}_t B(x, t, r_t) \geq \kappa r_t^3 = 2^{3/4}\kappa(1-t)^{3/2}$, contradicting (5.11) for $t \to -\infty$. ☐

So any 3 dimensional κ-solution $M \times (-\infty, 0]$ is either diffeomorphic to a spherical space form, to $\mathbb{R}^3$ or is homothetic to one of the following quotients of the round shrinking cylinder:

$$S^2 \times \mathbb{R},\ \ \mathbb{R}P^2 \times \mathbb{R}\ \text{ or }\ S^2 \tilde{\times} \mathbb{R}.$$

In the case in which a $M \times (-\infty, 0]$ is a *higher spherical space form* (hereby we mean a spherical space form that is not S^3 or $\mathbb{R}P^3$), we will show that $M \times (-\infty, 0]$ is already *homothetic* to the corresponding quotient of the round shrinking $S^3 \times (-\infty, 0]$. However, this result will not be needed in the subsequent chapters.

If a κ-solution is diffeomorphic to S^3, $\mathbb{R}P^3$ or $\mathbb{R}^3$, we cannot give a precise classification of its geometry. For example, it is not known if the round shrinking S^3 is the only κ-solution diffeomorphic to S^3. It is very reasonable that there is also a solution that looks more and more like an ellipsoid and approaches the geometry of a round cylinder around its center or a bowl-shaped solution on $\mathbb{R}^3$ around its tips for $t \to -\infty$. The analogon can be conjectured for $\mathbb{R}P^3$. Note that there is a κ-solution on $\mathbb{R}^3$ that is rotationally symmetric and asymptotically cylindrical in its end, called *Bryant's soliton* (see [Cho2, Ch 1] or [CLN, Sec 4.6]). Moreover, this solution is a steady gradient soliton. It is unknown if this is the only κ-solution on $\mathbb{R}^3$ up to homothety.

However, in the two unknown cases we will still be able to give give an approximate classification. At first, we have to analyze the global structure of 3 dimensional κ-solutions.

5.4.1 The asymptotic cone of a 3 dimensional κ-solution

In the proof of Proposition 5.1.3 we have already seen that the blow downs of a κ-solution do not smoothly converge to a smooth metric cone of the same dimension away from its tip. In dimension 3 even more is true:

Proposition 5.4.2. *Let $M \times (-\infty, 0]$ be a 3 dimensional κ-solution. Then the asymptotic cone of $M(0)$ is either a point, a ray or a line. If it is a point, the solution is compact. If it is a line, the solution is either homothetic to the round $\big(S^2 \times \mathbb{R}\big) \times (-\infty, 0]$ or to the round $\big(\mathbb{R}P^2 \times \mathbb{R}\big) \times (-\infty, 0]$.*

Proof. Let $p \in M$ and (C, p_∞) be the asymptotic cone of $M(0)$. If C is just a single point, $M(0)$ doesn't have rays hence it is compact (recall the construction of the asymptotic cone in the proof of Proposition 3.6.1).

Assume that (C, p_∞) is not just a ray starting in p_∞. We want to show that $M \times (-\infty, 0]$ is homothetic to the standard $\big(S^2 \times \mathbb{R}\big) \times (-\infty, 0]$ or to $\big(\mathbb{R}P^2 \times \mathbb{R}\big) \times (-\infty, 0]$. This will then imply that (C, p_∞) is a line. We can find two rays $\gamma, \sigma : [0, \infty) \to M(0)$ parameterized by arclength and starting in p such that

$$\lim_{s,t \to \infty} \widetilde{\sphericalangle} \gamma(t) p \sigma(s) = \alpha > 0. \tag{5.12}$$

Consider the universal covering κ-solution $\widetilde{M} \times (-\infty, 0]$. Lift p to get $\widetilde{p}$ and, starting in $\widetilde{p}$, lift γ and σ to get $\widetilde{\gamma}$ resp. $\widetilde{\sigma}$. Obviously the above identity is also true for $\widetilde{\gamma}$ and $\widetilde{\sigma}$, possibly for a different $\widetilde{\alpha} > \alpha$. If we could show that $\widetilde{M} \times (-\infty, 0]$ is homothetic to the round $(S^2 \times \mathbb{R}) \times (-\infty, 0]$ then $M \times (-\infty, 0]$ is homothetic to the standard Ricci flow on one of its quotients, i.e.

$$S^2 \times \mathbb{R}, \quad \mathbb{R}P^2 \times \mathbb{R}, \quad S^2 \widetilde{\times} \mathbb{R} = \big(S^2 \times \mathbb{R}\big) / (a, b) \sim (-a, -b)$$

or some compact quotient of $S^2 \times \mathbb{R}$. Since the asymptotic cone of $M(0)$ is more than a ray, we can exclude all cases except the the first two. This shows that it suffices to assume M to be simply connected.

From (5.12) and the fact that $\widetilde{\sphericalangle} \gamma(t) p \sigma(s)$ is decreasing in s and t we conclude that

$$\mathrm{dist}_0(\gamma(t), \sigma) \geq t \sin \alpha. \tag{5.13}$$

Since $t^2 S(p, 0) = \mathrm{dist}_0^2(p, \gamma(t)) S(p, 0) \to \infty$ we get by Corollary 5.2.6 that

$$t^2 S(\gamma(t), 0) \longrightarrow \infty. \tag{5.14}$$

For every $t \geq 0$ consider the rescaled pointed κ-solution

$$(M_t \times (-\infty, 0], (\gamma(t), 0)) = (\lambda_t(M \times (-\infty, 0]), (\gamma(t), 0)) \quad \text{where} \quad \lambda_t = S^{1/2}(\gamma(t), 0).$$

On $M_t \times (-\infty, 0]$ we have $S(\gamma(t), 0) = 1$ and so from (5.14) we get that $\mathrm{dist}_0(p, \gamma(t)) = \lambda_t t \to \infty$. By Proposition 5.2.8, there is a subsequence $t_k \to \infty$ such that we have smooth convergence of the $(M_{t_k} \times (-\infty, 0], (\gamma(t_k), 0))$ to a κ-solution $(M_\infty \times (-\infty, 0], (x_\infty, 0))$. Since $\gamma(t_k) \to x_\infty$ and $\gamma|_{[0,2t_k]}$ are minimizing segments whose length on $M_{t_k} \times (-\infty, 0]$ diverges in k, the segments subconverge to a line $\gamma_\infty \subset M_\infty(0)$ and thus $M_\infty \times (-\infty, 0] = (N \times \mathbb{R}) \times (-\infty, 0]$ where $N \times (-\infty, 0]$ is a 2 dimensional $\frac{1}{2}\kappa$-solution hence compact. Let $d := \mathrm{diam}_0 N$.

We have shown that for any $\varepsilon > 0$ there are ε-isometries

$$\Phi_k : N(0) \times (-2d, 2d) \longrightarrow \lambda_{t_k} M(0) \qquad \text{with} \qquad x_\infty \mapsto \gamma(t_k)$$

for large k. Set $U_k := \mathrm{Im}\, \Phi_k$. Using the simply connectedness of M we get as in the proof of Corollary 5.1.9 that U_k separates M into two components such that p and $\gamma|_{(t_k + 3d\lambda_k, \infty)}$ lie in different components for large k (if ε is chosen small enough). Thus the component of $M \setminus U_k$ that does not contain p, is noncompact for large k.

Since $\mathrm{diam}_0 U_k \leq \frac{6d}{\lambda_{t_k}} = \frac{6d}{t_k S^{1/2}(\gamma(t_k), 0)} t_k$, we conclude from (5.14) and (5.13) that for large k the ray σ does not hit U_k. Hence the component of $M \setminus U_k$ containing p is also noncompact. So $M(0)$ has two ends and thus splits off an $\mathbb{R}$ factor. This implies that $M \times (-\infty, 0] = (N \times \mathbb{R}) \times (-\infty, 0]$ where $N \times (-\infty, 0]$ is a simply connected 2 dimensional $\frac{1}{2}\kappa$-solution. By Theorem 5.3.5, $N \times (-\infty, 0]$ is homothetic to the round shrinking $S^2 \times (-\infty, 0]$. $\square$

5.4.2 ε-necks and ε-caps

As will become apparent later, an important property of κ-solutions is that they look locally either cylindrical with S^2 or $\mathbb{R}P^2$ as cross section or resemble a cap diffeomorphic to $\mathbb{B}^3$ or $\mathbb{R}P^3 \setminus \overline{B}^3$. We will make this precise:

Definition 5.4.3 (ε-neck). *Let $\varepsilon > 0$, (M, g) be a Riemannian manifold and $U \subset M$ be an open subset. We call U an* ε-neck *if there is a bijective ε-homothety $\Phi : S^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \to U$, where $S^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$ denotes the corresponding subset of the standard round cylinder. Moreover, we call $x \in U$ a* center of U *if $x \in \Phi(S^2 \times \{0\})$ for such a Φ.*

As for Ricci flows it will turn out to be useful to have control over the local geometry at earlier times as well.

Definition 5.4.4 (strong ε-neck). *Let $\varepsilon > 0$, $M \times I$ be a Ricci flow, $U \subset M$ an open subset and $J = [t_1, t_2] \subset I$ a closed subinterval. We say that $U \times J$ is a* strong ε-neck *if there is a scaling factor $\lambda > 0$ such that after parabolically rescaling the flow on $U \times J$ by the factor λ^{-1}, the time interval $\lambda^{-2} J = [\lambda^{-2} t_1, \lambda^{-2} t_2]$ has length $\lambda^{-2}(t_2 - t_1) = \varepsilon^{-2}$ and there is a (bijective) diffeomorphism $\Phi : S^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \to U$ that is an ε-isometry between the time t metric of the standard round cylinder on $S^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$ and the time $t + \lambda^{-2} t_2$ metric on $\lambda^{-1}(M \times J)$ for all $t \in [-\varepsilon^{-2}, 0]$.*
Moreover, we call $(x, t_2) \in U \times \{t_2\}$ a center of $U \times J$ *if $x \in \Phi(S^2 \times \{0\})$ for such a Φ.*

Observe that this definition differs slightly from the definition given in [Per1] since we require $\lambda^{-2}(t_2 - t_1)$ to be equal to ε^{-2} rather than 1. Choosing this convention will simplify the reasoning in subsection 6.1 and in the proof of Lemma 7.3.5.

Obviously, the final time slice of a strong ε-neck is an ε-neck. Particularly, every center of a strong ε-neck is also the center of an ε-neck. If U is an ε-neck and $\varepsilon' > \varepsilon$ then there is an open subset $U' \subset U$ that is an ε'-neck. Moreover, for any center $x \in U$ we can choose

U' such that x is still a center of U'. Analogous statements hold true for strong ε-necks. If $U \subset M$ is an ε-neck and λ the scaling factor of one of the ε-homotheties Φ in Definition 5.4.3, then we can estimate the scalar curvature S on U by $\frac{1}{\mu(\varepsilon)}\lambda^{-2} < S < \mu(\varepsilon)\lambda^{-2}$ if ε is small enough (e.g. smaller than $\frac{1}{2}$). The constant μ depends only on ε and $\mu \to 1$ for $\varepsilon \to 0$. Again, we have analogous statements for strong ε-necks.

Let $U \subset M$ and $\Phi : S^2 \times (A, B) \to U$ be a (bijective) diffeomorphism. We call an embedding $\iota : S^2 \to U$ *parallel* to Φ if the map $\mathrm{pr}_1 \circ \Phi^{-1} \circ \iota : S^2 \to S^2$ is a diffeomorphism (here $\mathrm{pr}_1 : S^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \to S^2$ denotes the projection onto the first factor). If $U \subset M$ is an ε-neck, we call ι parallel to the neck U if it is parallel to every $\Phi : S^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \to U$ from Definition 5.4.3.

Now consider two maps $\Phi_{1/2} : S^2 \times (-A, A) \to U_{1/2}$ and assume that the image of the central cross-section $\Phi_1(S^2 \times \{0\})$ lies in $\Phi_2(S^2 \times (0, A))$ resp. $\Phi_2(S^2 \times (-A, 0))$ and is parallel to Φ_2. Then it is easy to see that we can glue the maps Φ_1 and Φ_2 together to get a surjective local diffeomorphism $\Phi_3 : S^2 \times (B, C) \to U_1 \cup U_2$ that coincides at its ends with $\Phi_1|_{S^2\times(-A,0)}$ or $\Phi_1|_{S^2\times(0,A)}$ as well as $\Phi_2|_{S^2\times(-A,0)}$ or $\Phi_2|_{S^2\times(0,A)}$ up to translations on $S^2 \times \mathbb{R}$.

We will see that in the case in which U_1 and U_2 are ε-necks, the parallelity is already implied if ε is small enough. For this we will introduce a constant ε_0 that we will decrease in the course of the exposition to allow conclusions involving ε-necks. Observe that ε_0 is independent of any other constant introduced subsequently.

Lemma 5.4.5. *There is an $\varepsilon_0 > 0$ such that for $\varepsilon_1, \varepsilon_2 < \varepsilon_0$ the following holds: If $U_1, U_2 \subset M$ are two ε_1- resp. ε_2-necks in a Riemannian manifold (M, g), then for any bijective ε_1-homothety $\Phi_1 : S^2 \times (-\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_1}) \to U_1$ the cross-sections $\Phi_1(S^2 \times \{a\})$ that lie in U_2 are parallel to the ε_2-neck U_2.*

Proof. Let $\Phi_2 : S^2 \times (-\frac{1}{\varepsilon_2}, \frac{1}{\varepsilon_2}) \to U_2$ be a bijective ε_2-homothety and $a \in (-\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_1})$ such that $\Sigma := \Phi_1(S^2 \times \{a\}) \subset U_2$. Consider the corresponding embedding $\iota = \Phi|_{S^2\times\{a\}}$.

Let X_1 resp. X_2 be unit vector fields on $S^2 \times (-\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_1})$ resp. $S^2 \times (-\frac{1}{\varepsilon_2}, \frac{1}{\varepsilon_2})$ in the direction of the second factor. If ε_0 is sufficiently small, we can apply the following reasoning: For any $x \in \Sigma$ the vectors $((\Phi_1)_* X_1)_x$ and $((\Phi_2)_* X_2)_x$ are close enough to the eigenspaces of Ric corresponding to the smallest eigenvalue that we may assume $\sphericalangle \left(((\Phi_1)_* X_1)_x , ((\Phi_2)_* X_2)_x \right) < \frac{\pi}{3}$ if we choose the orientations of X_1 and X_2 appropriately. Moreover, we may assume that $\sphericalangle \left((\Phi_1)_* X_1, T\Sigma \right), \sphericalangle \left((\Phi_2)_* X_2, T\Sigma \right) < \frac{\pi}{3}$ along Σ. So $\Phi_2^{-1}\Sigma$ is transversal to X_2 and we conclude that $\Phi_2^{-1} \circ \iota$ is a local diffeomorphism hence a diffeomorphism. $\square$

Eventually, we explain what we mean by a cap. We will not need the "strong" version here. Note that the main point in the following definition is that the part whose geometry is not controlled by ε-isometries can be geometrically bounded.

Definition 5.4.6 (**(ε, E)-cap**). *Let $\varepsilon > 0$, $E < \infty$ and (M, g) be a Riemannian manifold. Consider an open set $U \subset M$ and suppose that* $\operatorname{diam}^2 US(x) < E^2$ *for any $x \in U$ and $\frac{1}{E^2}S(x) \leq S(y) \leq E^2 S(x)$ for any $x, y \in U$. Furthermore, assume that U is either diffeomorphic to $\mathbb{B}^3$ or $\mathbb{R}P^3 \setminus \overline{\mathbb{B}}^3$ and that there is a compact set $K \subset U$ such that $U \setminus K$ is a ε-neck. Then U is called an* (ε, E)-cap *or if we don't want to be so precise, an* ε-cap. *If $x \in K$ for such a K, then we say that x is a center of U.*

Observe that every center of an (ε, E)-cap is also the center of an (ε', E')-cap if $\varepsilon' \geq \varepsilon$ and $E' \geq E$.

We will now use the results obtained so far to prove the following Proposition:

Proposition 5.4.7. *There is an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ the following holds: Let (M, g) be a Riemannian manifold and $W \subset M$ such that any point $w \in W$ is the center* *of an ε-neck or an ε-cap. Then W is covered by disjoint connected open sets $N_k \subset M$ diffeomorphic to*

$$S^2 \times \mathbb{R},\ \ \mathbb{B}^3,\ \ \mathbb{R}P^3 \setminus \overline{\mathbb{B}}^3,\ \ S^3,\ \ S^2 \times S^1,\ \ S^2 \tilde{\times} S^1,\ \ \mathbb{R}P^3\ \ \text{or}\ \ \mathbb{R}P^3 \# \mathbb{R}P^3.$$

Moreover, the N_k are covered by the given ε-necks and -caps.

If every point $w \in W$ is the center of an ε-neck, then the N_k are diffeomorphic to $S^2 \times \mathbb{R}$, $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$.

Proof. Let W' be the set of all centers of ε-necks in W plus the centers of the ε-necks corresponding to the given ε-caps. Consider a collection $\mathcal{N}$ of pairs (U, w) of ε-necks $U \subset M$ and their centers w, such that for every $w \in W'$ there is a $U \subset M$ with $(U, w) \in \mathcal{N}$.

Let $(U_0, w_0), (U_1, w_1) \in \mathcal{N}$ and assume that $\frac{1}{10\varepsilon} < \operatorname{dist}(w_0, w_1) < \frac{1}{2\varepsilon}$. Choose $\Phi_{0/1} : (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \to U_{0/1}$ as in Definition 5.4.3. For ε_0 sufficiently small this implies that the cross-section $\Sigma := \Phi_1(S^2 \times \{0\})$ of U_1 lies in U_0. Since we have a lower bound on the distance between the points w_0 and w_1, we may assume (for small enough ε_0) that Σ is contained in $\Phi_1(S^2 \times (0, \frac{1}{\varepsilon}))$ (possibly after flipping Φ_1 on its $(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$ factor). Thus by Lemma 5.4.5 and the preceding discussion we can glue the maps Φ_0 and Φ_1 together to obtain a local diffeomorphism $\Psi : S^2 \times (A, B) \to M$ that coincides at one end with $\Phi_0|_{S^2 \times (-\frac{1}{\varepsilon}, 0)}$ and at the other with $\Phi_1|_{S^2 \times (0, \frac{1}{\varepsilon})}$ (if we choose the right orientation for Φ_1) up to translations on $S^2 \times \mathbb{R}$.

Assume that there is an element $(U_2, w_2) \in \mathcal{N}$ with $\frac{1}{10\varepsilon} < \operatorname{dist}(w_1, w_2) < \frac{1}{2\varepsilon}$ and $w_2 \in \Phi_1(S^2 \times (0, \frac{1}{\varepsilon}))$. Choose a corresponding $\Phi_2 : (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \to U_2$. Analogous to the discussion in the preceding paragraph we may assume that $\Sigma' := \Phi_2(S^2 \times \{0\}) \subset \Phi_1(S^2 \times (0, \frac{1}{\varepsilon}))$. Repeat the preceding argument with Φ_0 resp. Φ_1 replaced by Φ_1 resp. Φ_2. Since Ψ coincides with $\Phi_1|_{S^2 \times (0, \frac{1}{\varepsilon})}$ at one end (up to a translation on $S^2 \times \mathbb{R}$), we can glue Ψ and Φ_2 together to produce a local diffeomorphism $\Psi' : (A', B') \to U_0 \cup U_1 \cup U_2$ that coincides at its ends with $\Phi_0|_{S^2 \times (-\frac{1}{\varepsilon}, 0)}$ and $\Phi_2|_{S^2 \times (0, \frac{1}{\varepsilon})}$ (if Φ_2 has the appropriate orientation) up to translations on $S^2 \times \mathbb{R}$.

We can repeat this process as long as we find $(U_i, w_i) \in \mathcal{N}$ for $i \to \infty$ with the claimed properties and successively extend Ψ at one end. If the process stops, we decrease the width of the last ε-neck such that the corresponding end is contained in a ball of radius $\frac{1}{5\varepsilon}$ around the last w_i. Points of W' that have lain in the image of Ψ will then still be covered. We proceed in an analogous way to extend Ψ at the other end.

We have constructed a local diffeomorphism $\Psi : S^2 \times (A, B) \to N$ that may still have self intersections. But since the ends of Ψ (if they exist) are covered by balls of radius $\frac{1}{5\varepsilon}$ around some points w_{-i_1} and $w_{i_2} \in W$, we can make sure by analogous cutting and glueing arguments that in this case the image of Ψ is diffeomorphic to $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$. Set $N' := \operatorname{Im} \Psi$.

We can apply the glueing process described so far for any starting point $w_0 \in W'$. This way we produce a collection of sets N' each of diffeomorphism type $S^2 \times \mathbb{R}$, $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$ that cover W'. By the way we choped off ends, we can ensure that for any two such N'_1, N'_2 either $N'_1 \cap W' = N'_2 \cap W'$ or $N'_1 \cap N'_2 = \emptyset$. Hence we can choose a sequence N'_k of disjoint subsets that still cover W'. Consider the points in $W \setminus W'$. These are centers of ε-caps whose corresponding ε-necks have centers that lie in $\bigcup_k N'_k$. It is easy to see that we can now glue any ε-cap into a corresponding N'_k and hereby produce N_k of the claimed diffeomorphism types. $\square$

A useful property of ε-necks $U \subset M$ is that there are geodesics (or hinges of large angles) that cross U. This makes them distinguishable from ε-caps, a fact that will prove to be useful for us lateron.

Lemma 5.4.8. *There is an $\varepsilon_0 > 0$ and an $\alpha > 0$ such that for $\varepsilon < \varepsilon_0$ the following holds: Let (M, g) be a Riemannian manifold and $x \in M$ center of an ε-cap U. Then there is no minimizing geodesic segment γ passing through x whose endpoints lie outside of U. If M is complete and has nonnegative sectional curvature, then for any two points y_1, y_2 outside of U we have $\widetilde{\sphericalangle} y_1 x y_2 < \pi - \alpha$.*

Proof. The first part is obvious. For the second part connect x and y_1, y_2 by minimizing geodesics γ_1 resp. γ_2 and choose $y'_{1/2} \in \gamma_{1/2} \cap \partial U$. By Toponogov's Theorem 1.3.1 we conclude that $\widetilde{\sphericalangle} y_1 x y_2 \leq \widetilde{\sphericalangle} y'_1 x y'_2$. Let $U' \subset U$ be an ε-neck as in the Definition 5.4.6 of ε-caps and λ its scaling factor. Then (for small enough ε_0) we have $\operatorname{dist}(y'_1, y'_2) < 3\pi\lambda$ and $\operatorname{dist}(x, y'_1), \operatorname{dist}(x, y'_2) > \frac{1}{\varepsilon}\lambda$ since x lies outside U'. This gives us an upper bound for $\widetilde{\sphericalangle} y'_1 x y'_2$. ☐

5.4.3 The geometry of 3 dimensional κ-solutions

We will now give the final classification result on 3 dimensional κ-solutions. Since the only non-orientable κ-solution is homothetic to the round shrinking $(\mathbb{R}P^2 \times \mathbb{R}) \times (-\infty, 0]$, we will restrict ourselves from now on to orientable κ-solutions. For an $\varepsilon > 0$ and a κ-solution $M \times (-\infty, 0]$ define $M_\varepsilon \subset M$ to be the set of points in $M(0)$ that are centers of strong ε-necks.

Lemma 5.4.9. *For any angle $\varphi \in (0, \pi]$ and any $\varepsilon > 0$ there is a $C_\kappa(\varphi, \varepsilon)$ such that the following holds:*
If $M \times (-\infty, 0]$ is an orientable 3 dimensional κ-solution and $p, q, q' \in M$ are points such that

$$\widetilde{\sphericalangle} q p q' \geq \varphi \qquad \textit{and} \qquad \operatorname{dist}_0^2(p, q) S(p, 0), \ \ \operatorname{dist}_0^2(p, q') S(p, 0) \geq C,$$

then $p \in M_\varepsilon$.

Proof. The assumptions of the lemma are scale invariant, so we can assume that $S(p, 0) = 1$. Fix some $\varphi, \varepsilon, \kappa > 0$ and assume that there is no such C. Let $M_k \times (-\infty, 0]$ and $p_k, q_k, q'_k \in M_k$ be sequences of counterexamples satisfying $S(p_k, 0) = 1$, $\widetilde{\sphericalangle} q_k p_k q'_k \geq \varphi$ and $\operatorname{dist}_0(p_k, q_k), \operatorname{dist}_0(p_k, q'_k) \to \infty$ but not $p_k \in (M_k)_\varepsilon$. Connect the p_k with the q_k resp. q'_k by time 0 minimizing geodesics γ_k resp. γ'_k parameterized by arclength. From Toponogov's Theorem 1.3.1 we get that

$$\widetilde{\sphericalangle} \gamma_k(s) p_k \gamma'_k(s') \geq \varphi \qquad \text{for any } s, s' > 0. \tag{5.15}$$

By Proposition 5.2.8 a subsequence of the $(M_k \times (-\infty, 0], (p_k, 0))$ converges to an orientable κ-solution $(M_\infty \times (-\infty, 0], (p_\infty, 0))$. Moreover, the geodesics γ_k resp. γ'_k subconverge to rays γ_∞ resp. γ'_∞. By (5.15) for any $s, s' > 0$ we have $\widetilde{\sphericalangle} \gamma_\infty(s) p \gamma'_\infty(s') \geq \varphi$. So the asymptotic cone of $M_\infty \times (-\infty, 0]$ consists of more than just one ray and by Proposition 5.4.2 the κ-solution $M_\infty \times (-\infty, 0]$ has to be isometric to the round shrinking cylinder $(S^2 \times \mathbb{R}) \times (-\infty, 0]$. This gives the desired contradiction. ☐

Lemma 5.4.10. *For any $\varepsilon > 0$ there is an $\alpha_\kappa(\varepsilon) < \infty$ such that the following holds:*
Let $M \times (-\infty, 0]$ be an orientable 3 dimensional κ-solution and $x, y \in M \setminus M_\varepsilon$ such that $\operatorname{dist}_0^2(x, y) S(x, 0) \geq \alpha$. Let $z \in M$. Then
(i) $\operatorname{dist}_0^2(x, z) S(x, 0) < \alpha$ *or*
(ii) $\operatorname{dist}_0^2(y, z) S(y, 0) < \alpha$ *or*
(iii) $\operatorname{dist}_0(x, z), \ \operatorname{dist}_0(y, z) < \operatorname{dist}_0(x, y)$ *and* $z \in M_\varepsilon$.
As a consequence, M is compact.

Observe that the assumptions and the hypothesis of the lemma are scale invariant and that there is a hidden symmetry when talking about the normalized distances (compare Corollary 5.2.6), e.g. the lemma stays true if we replace $\operatorname{dist}_0^2(x, y) S(x, 0)$ by $\operatorname{dist}_0^2(x, y) S(y, 0)$.

Proof. Set $\alpha_\kappa(\varepsilon) := \max(C_2(C_\kappa(\frac{\pi}{3},\varepsilon),\kappa), C_\kappa(\frac{\pi}{3},\varepsilon))$ where we use C_2 from Corollary 5.2.6 and C from Lemma 5.4.9. Assume that (i) and (ii) do not hold. Corollary 5.2.6 gives us

$$\operatorname{dist}_0^2(x,z)S(z,0), \ \operatorname{dist}_0^2(y,z)S(z,0), \ \operatorname{dist}_0^2(x,y)S(y,0) \geq C_\kappa\left(\frac{\pi}{3},\varepsilon,\kappa\right).$$

From Lemma 5.4.9 applied to x and y we conclude that $\widetilde{\sphericalangle} yxz, \widetilde{\sphericalangle} zyx < \frac{\pi}{3}$ and the first part of (iii) follows. So we must have $\widetilde{\sphericalangle} yzx > \frac{\pi}{3}$ and Lemma 5.4.9 applied to z yields the second part of (iii). $\square$

Together with Proposition 5.4.7 this result implies that if we choose ε_0 small enough, then for $0 < \varepsilon < \varepsilon_0$ and $\alpha = \alpha_\kappa(\varepsilon)$ any orientable κ-solution $M \times (-\infty, 0]$ falls in (at least) one of the following categories:

(A) $M_\varepsilon = M$, hence M is diffeomorphic to either $S^2 \times \mathbb{R}$, $S^2 \times S^1$ or $S^2 \widetilde{\times} S^1$. By Lemma 5.4.1 the latter two cases are impossible. So $M \times (-\infty, 0]$ is homothetic to the round shrinking cylinder $(S^2 \times \mathbb{R}) \times (-\infty, 0]$.

(B) There is a point $x \in M \setminus M_\varepsilon$ and $M = M_\varepsilon \cup B$ where $B := B_{\alpha S^{-1/2}(x,0)}(x, 0)$. So $M \setminus B$ is covered by disjoint open sets N_k diffeomorphic to $S^2 \times \mathbb{R}$ which are covered by ε-necks themselves.

 Assume that for some k the set N_k is bounded. Then N_k has two boundary components which are both contained in B. Suppose first that there is an ε-neck $U \subset N_k$ that doesn't hit B. Since B is connected, we can find a loop $b : S^1 \to M$ that has intersection number 1 with a cross-section Σ of U. So $[b] \in \pi_1 M$ has infinite order and thus any lift of Σ in the universal cover $\widetilde{M}$ separates $\widetilde{M}$ into two noncompact parts. We conclude that in this case $\widetilde{M}$ has two ends, so $\widetilde{M} \times (-\infty, 0]$ is homothetic to the round shrinking cylinder and by Lemma 5.4.1 $M \times (-\infty, 0]$ must homothetic to the round shrinking $(S^2 \widetilde{\times} \mathbb{R}) \times (-\infty, 0]$, contradicting the boundedness of N_k.

 On the other hand, if every ε-neck $U \subset N_k$ intersects B, we deduce that for any $y \in N_k$ there is some $y' \in B$ such that y and y' lie in a common ε-neck. Hence by Corollary 5.2.5 we get $\operatorname{dist}_0(y, y') < \frac{3}{\varepsilon} S^{-1/2}(y', 0) < \frac{3}{\varepsilon} C_{\kappa,3}^{1/2}(\alpha) S^{-1/2}(x, 0)$ for ε_0 small enough. We conclude that N_k is contained in a ball B' of radius $C' S^{-1/2}(x, 0) := (\alpha + \frac{3}{\varepsilon} C_{\kappa,3}^{1/2}(\alpha)) S^{-1/2}(x, 0)$ around x.

 Thus, the N_k which are not contained in B' are unbounded. It is easy to see that there is not more than one such N_k since otherwise M would have two ends and $M \times (-\infty, 0]$ must be homothetic to round cylindrical flow. Analogously, such an N_k cannot be unbounded in both directions.

 Suppose first that $M \neq B'$, so there is exactly one unbounded N_k. By lifting N_k to the universal cover, we get that if $\pi_1 M \neq 1$, the κ-solution $M \times (-\infty, 0]$ must be homothetic to the round $(S^2 \widetilde{\times} \mathbb{R}) \times (-\infty, 0]$. If, however, $\pi_1 M = 1$, then M can only be diffeomorphic to $\mathbb{R}^3$ and by Alexander's Theorem (see [Hat, Ch 1]) the cross-sections of N_k bound 3-balls containing x.

 Finally, if $M = B'$, we have $\operatorname{diam}_0 M < 2C' S^{-1/2}(x, 0)$. Observe that Corollary 5.2.5 implies for any $x' \in M$ that $S(x', 0) > C_{\kappa,3}^{-1} S(x, 0)$. So for

$$C'' := 2 C_{\kappa,3}^{1/2}(C') C'$$

 we get $\operatorname{diam}_0 M < C'' S^{1/2}(x', 0)$.

(C) There are two points $x_1, x_2 \in M \setminus M_\varepsilon$ such that $M = M_\varepsilon \cup B_1 \cup B_2$ with $B_{1/2} := B_{\alpha S^{-1/2}(x_{1/2},0)}(x_{1/2}, 0)$ and M is compact. So M is a spherical space form.

Set $B'_{1/2} := B_{C'S^{-1/2}(x_{1/2},0)}(x_{1/2},0)$ where we use the constant C' from (B). Assume for the moment without loss of generality that $S(x_1,0) \leq S(x_2,0)$. If B'_1 and B'_2 intersect, we conclude using Lemma 5.4.10 that $\operatorname{diam}_0 M \leq 3C'S^{-1/2}(x_1,0)$. So, analogously to the last paragraph, for

$$C''' := 3C_{\kappa,3}^{1/2}(3C')C'$$

we have $\operatorname{diam}_0 M < C'''S^{-1/2}(x',0)$ for any $x' \in M$. Assume from now on B'_1 and B'_2 to be disjoint.

By Proposition 5.4.7 the manifold $M(0)$ is covered by B_1, B_2 and disjoint open sets $N_k \subset M$ diffeomorphic to $S^2 \times \mathbb{R}$ which are covered by ε-necks themselves. Analogously to the case (B) we conclude that either the two boundary components of every N_k lie in one of the B_1 resp. B_2 each or N_k is contained in $B'_1 \cup B'_2$.

Suppose that N_{k_1}, N_{k_2} have the property that their boundary components lie in B_1 resp. B_2 each. Then as in case (B), there is a loop $b : S^1 \to M$ that has intersection number 1 with a cross-section Σ of N_{k_1} or N_{k_2} and the universal covering flow is homothetic to round cylindrical flow contradicting the compactness of M.

So M is covered by B'_1, B'_2 and exactly one open set $N \subset M$ diffeomorphic to $S^2 \times \mathbb{R}$. Let $\Sigma \subset N$ be a cross-sectional 2-spheres of N. The preimage of Σ under the universal covering map $\pi : S^3 \to M$ consists of disjoint copies of 2-spheres. So one component of $\pi^{-1}\Sigma$ bounds a closed imedded ball that is disjoint from the other components (here we applied Alexander's Theorem). Since the group of covering transformations acts transitively on the components of $\pi^{-1}\Sigma$, this has to be true for all components. So $\Sigma \subset M$ also bounds an embedded ball $D \subset M$. Without loss of generality we may assume $x_1 \in D$. Furthermore, after possibly enlarging D we can assume $\Sigma \subset B'_2$. Then obviously, $B'_1 \subset D$.

Note that if M is diffeomorphic to S^3, every cross-section of N bounds embedded balls on both sides and if M is diffeomorphic to $\mathbb{R}P^3$, then Σ bounds an embedded ball on one side and an embedded $\mathbb{R}P^3 \setminus \mathbb{B}^3$ on the other side.

Assume now that M is not diffeomorphic to S^3 or $\mathbb{R}P^3$, i.e. M is a higher spherical space form. Then $|\pi_1 M| > 2$. Consider the points $\pi^{-1}x_1 \subset \pi^{-1}D$. These points cannot be centers of strong ε-necks in $\mathcal{M}$ since such necks have to lie in $\pi^{-1}B'_1 \subset \pi^{-1}D$ and the lifts of D are disjoint and project one-to-one to D under the covering map. But this contradicts Lemma 5.4.10 applied to the covering flow $\widetilde{M} \times (-\infty,0]$, since for any two $\widetilde{x}_1, \widetilde{x}'_1 \in \pi^{-1}x_1$ we have $\operatorname{dist}_0^2(\widetilde{x}_1, \widetilde{x}'_1)S(\widetilde{x}_1,0) > \alpha$. So in the case $B'_1 \cap B'_2 = \emptyset$ the manifold M cannot be a higher spherical space form.

The preceding discussion implies:

Theorem 5.4.11. *For any $\varepsilon > 0$ there is an $E_\kappa(\varepsilon)$ such that for any orientable 3 dimensional κ-solution $M \times (-\infty,0]$ and any $x \in M$ one of the following cases applies:*
(a) x is the center of a strong ε-neck
(b) x is the center of an (ε,E)-cap,
(c) M is a higher spherical space form
(d) $M \times (-\infty,0]$ is homothetic to the round shrinking $(\mathbb{R}P^2 \times \mathbb{R}) \times (-\infty,0]$.
Moreover, whenever (c) applies, $\operatorname{diam}_0 M < ES^{-1/2}(x,0)$.

In case (c) we even have more:

Theorem 5.4.12. *Let $M \times (-\infty,0]$ be a compact 3 dimensional κ-solution that is not diffeomorphic to S^3 or $\mathbb{R}P^3$. Then $M \times (-\infty,0]$ is homothetic to a quotient of the round shrinking $S^3 \times (-\infty,0]$.*

Proof. For a compact 3 dimensional Riemannian manifold (N, g) of positive Ricci curvature denote

$$P(N) := \max\{c \in \mathbb{R} : \mathrm{Ric} \geq cS\}.$$

Obviously, P is scale-invariant and $0 < P(N) \leq \frac{1}{3}$. Furthermore, $P(N) = \frac{1}{3}$ if and only if N is homothetic to a quotient of the round sphere S^3. As we know from Corollary 2.5.8, the quantity P is nondecreasing under the Ricci flow and if it is locally constant, N is homothetic to a quotient of the round sphere S^3.

Now consider the collection $\mathcal{F}$ of all pointed compact κ-solutions $(M \times (-\infty, 0], (x, 0))$ with $S(x, 0) = 1$ that are not diffeomorphic to S^3 or $\mathbb{R}P^3$. These solutions must be diffeomorphic to a spherical space form and by Theorem 5.4.11 we have $\mathrm{diam}_0 M < ES^{-1/2}(x, 0)$. We will apply an analogous reasoning as in the proof of Theorem 5.3.5: $\mathcal{F}$ is compact with respect to Gromov-Hausdorff limits and $F_0 : \mathcal{F} \to \mathbb{R}$ is continuous. Furthermore, any κ-solution in $\mathcal{F}$ has positive Ricci curvature since it is not diffeomorphic to a metric quotient of $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ (recall the discussion in section 1.4). So there is a κ-solution $(M \times (-\infty, 0], (x, 0)) \in \mathcal{F}$ for which P_0 attains its mininum. But since P is nondecreasing under the Ricci flow, $P_t(M)$ has to be constant in time and thus $P(M) \equiv \frac{1}{3}$. This implies that $P_0 = \frac{1}{3}$ on $\mathcal{F}$ and all elements of $\mathcal{F}$ are round. $\square$

5.4.4 A universal κ_0

In dimension 3 we even have the following

Theorem 5.4.13. *There is a $\kappa_0 > 0$ such that any 3 dimensional κ-solution that is not diffeomorphic to a higher spherical space form, is in fact a κ_0-solution.*

A proof can be found in [MT, Ch 9.5] or [KL, Proposition 49.1]. We will only give a short sketch of the proof since some non-basic analytical tools are needed to carry out the details:

Consider a 3-dimensional κ-solution $M \times (-\infty, 0]$ not diffeomorphic to a higher spherical space form and choose a basepoint (x_0, t_0). By the results of chapter 4 for any $\tau > 0$ there is a point $q(\tau) \in M$ such that the l-distance between (x_0, t_0) and $(q(\tau), t_0 - \tau)$ does not exceed $\frac{3}{2}$. Now, we can show that there is a sequence $\tau_k \to \infty$ such that the pointed parabolically rescaled Ricci flows $(\tau_k^{-1/2}(M \times [t_0 - 2\tau_k, t_0 - \tau_k]), (x_k, t_0 - \tau_k))$ converge to a solution $(M_\infty \times [-2, -1], (x_\infty, -1))$ that is a non-flat gradient shrinking soliton with bounded curvature whose time slices are κ-noncollapsed. This soliton is called the *asymptotic soliton*. Note that this is the technically most demanding part of the proof.

By the results of [Ham1] or a similar argument as in the proof of Theorem 5.4.12 we can conclude that if $M_\infty \times [-2, -1]$ is compact, it must be homothetic to the round $S^3 \times [-2, -1]$ or $\mathbb{R}P^3 \times [-2, -1]$. If it is noncompact, we can show (see [KL, Lemma 50.1]) that $M_\infty(-1)$ cannot have positive sectional curvature everywhere. So there must be a point $y \in M_\infty$ and a plane $\pi \subset T_y M_\infty$ such that $K_{-1}(\pi) = 0$ and using the strong maximum principle, it is possible to show that the universal covering solution $\widetilde{M}_\infty \times [-2, -1]$ splits as $(N \times \mathbb{R}) \times [-2, -1]$ where $N \times [-2, -1]$ is a 2 dimensional gradient shrinking soliton whose time slices are $\frac{1}{2}\kappa$-noncollapsed. By Theorem 5.3.5 we can show that $N \times [-2, -1]$ is homothetic to the round $S^2 \times [-2, -1]$. So $M_\infty \times [-2, -1]$ is homothetic to one of the following round solutions: $(S^2 \times \mathbb{R}) \times [-2, -1]$, $(S^2 \widetilde{\times} \mathbb{R}) \times [-2, -1]$ or $(\mathbb{R}P^2 \times \mathbb{R}) \times [-2, -1]$.

We conclude that there is a universal $\kappa_0' > 0$ such that $M_\infty(0)$ is always κ_0'-noncollapsed. Now we can use the methods developed in the proof of the No Local Collapsing Theorem 4.2.4 to show that $M \times (-\infty, 0]$ is κ_0-noncollapsed in (x_0, t_0) for a κ_0 that only depends on κ_0'.

Chapter 6

Canonical neighborhoods

In this chapter we will geometrically characterize regions of large curvature in arbitrary 3 dimensional Ricci flows. We will show that these regions are geometrically close to κ-solutions and thus essentially strong necks or caps. Our aim is to prove the Canonical Neighborhood Theorem 6.3.2 on page 73 which is a deeper result than Theorem 5.0.1 in the sense that we can already characterize the local geometry if the scalar curvature $S(x,t)$ at some point (x,t) is large enough compared to some universal parameters. Previously, we needed to make a very special choice for (x,t) in order to do this. In chapter 7 we will use a somewhat generalized version of the Canonical Neighborhood Theorem to perform surgeries in an appropriate way.

6.1 Definitions

Fix κ_0 from Theorem 5.4.13 and accordingly choose $\eta = \eta_{\kappa_0,3}$ from Corollary 5.2.9. In the following we will frequently express that the neighborhood of some points of large curvature carry a certain standardized approximate geometry on a local scale. For this we introduce the following phrase:

Definition 6.1.1 (Canonical neighborhood assumptions). *Let $r, \varepsilon > 0$ and $E < \infty$. A point (x,t) in a Ricci flow $M \times I$ is said to* satisfy the canoncial neighborhood assumptions $\mathrm{CNA}(r,\varepsilon,E)$ *if either $S(x,t) \leq \frac{1}{r^2}$ or*
(A) $\|\nabla S^{-1/2}(x,t)\| < \frac{1}{2}\eta + \varepsilon$ and $|\partial_t S^{-1}(x,t)| < \left(\frac{1}{2}\eta + \varepsilon\right)^2$,
(B) $M(t)$ is $(\kappa_0 - \varepsilon)$-noncollapsed in x and
(C) (x,t) is the center of a strong ε-neck or an (ε, E)-cap.

Observe that we have chosen the definition such that if $(x,t) \in M \times [0,T]$ satisfies the canonical neighborhood assumptions $\mathrm{CNA}(r,\varepsilon,E)$ then there is a neighborhood $U \subset M \times I$ in spacetime around (x,t) whose points are $\mathrm{CNA}(\frac{1}{2}r, 2\varepsilon, 2E)$ (if ε is smaller than a certain universal constant what we will always assume in the course of this exposition). In addition, if $(x_k,t_k) \in M_k \times I_k$ is a sequence of points in a sequence of Ricci flows which converges to some Ricci flow $M_\infty \times I_\infty$ such that $(x_k,t_k) \to (x_\infty,t_\infty)$, then we can conclude: if all (x_k,t_k) satisfy the canonical neighborhood assumptions $\mathrm{CNA}(r,\varepsilon,E)$, then (x_∞,t_∞) is $\mathrm{CNA}(\frac{1}{2}r, 2\varepsilon, 2E)$. And vice versa: If (x_∞,t_∞) satisfies the canonical neighborhood assumptions $\mathrm{CNA}(r,\varepsilon,E)$ then (x_k,t_k) is $\mathrm{CNA}(\frac{1}{2}r, 2\varepsilon, 2E)$ for large k.

In order to control the extent to which negative sectional curvature can occur in a Ricci flow, we define (compare the Hamilton-Ivey pinching in section 2.9)

Definition 6.1.2. *Let $M \times [T_1,T_2]$ be a Ricci flow, $(x,t) \in M \times [T_1,T_2]$ and $\varphi > 0$. We say that the curvature at $(x,t) \in M \times [T_1,T_2]$ ($t > -\varphi^{-1}$) is* φ-positive *if there is an*

$X > 0$ such that the sectional curvature $K(x,t) \geq -X$ as well as

$$S(x,t) \geq -\frac{3}{\varphi^{-1}+t} \qquad \textit{and} \qquad S(x,t) \geq X(\log X + \log(\varphi^{-1}+t) - 3).$$

We say that a Riemannian manifold M has φ-positive curvature at time t *if $M \times \{t\}$ has φ-positive curvature. In the case $t = 0$ we simply say that M has* φ-positive curvature.

Note that if the curvature at (x,t) is φ-positive and if $\varphi' > \varphi$, then it is also φ'-positive. Furthermore, if the curvature at (x,t) is φ-positive for all $\varphi > 0$, then the sectional curvature at (x,t) is nonnegative. This fact implies that a limit of Ricci flows with φ_k-positive curvature has nonnegative sectional curvature if $\varphi_k \to 0$.

It is important to note that the property above depends on the time t. If the curvature in $(x,t) \in M \times [T_1, T_2]$ is φ-positive, then the curvature of $(x, t - t_0)$ in the shifted Ricci flow $M \times [T_1 - t_0, T_2 - t_0]$ is $(\varphi^{-1} + t_0)^{-1}$-positive. In addition, if we parabolically rescale $M \times [T_1, T_2]$ by the factor λ to get $\lambda(M \times [T_1, T_2]) = M' \times [\lambda^2 T_1, \lambda^2 T_2]$, the curvature at the corresponding point $(x', \lambda^2 t)$ is $\lambda^{-2}\varphi$-positive.

From the Hamilton-Ivey pinching (Theorem 2.9.1) we get that if M is compact and the curvature is φ-positive at time T_1, then this property is preserved by the Ricci flow, i.e. the curvature is φ-positive on all time slices.

Only in this chapter we want to broaden the definition of strong ε-necks since we will later have to deal with conventional Ricci flows that arise as time pieces of Ricci flows with surgery: Let $M \times I$ be a Ricci flow, $U \subset M$ and $J = [t_1, t_2] \subset \mathbb{R}$ a time interval so that $t_2 \in I$. We will say that $U \times J$ is a *strong ε-neck* if there is a Ricci flow on $U \times J$ that coincides with the Ricci flow on $M \times I$ on $U \times J \cap M \times I$ which is a a strong ε-neck. Note that for $I = \{t\}$ we have defined strong ε-necks for Riemannian manifolds. By the local Shi estimates (see Theorem 2.6.2) we immediately get bounds for the curvature derivatives $\nabla^l R$ on $U \times J$ (note that these bounds deteriorate towards the boundary resp. the first time slice). We say that $U \times J$ is a *strong ε-neck with nonnegative sectional curvature* if we can choose the Ricci flow on $U \times J$ such that the sectional curvature is nonnegative. Analogously we define *strong ε-necks with φ-positive curvature.* We also adapt the definition of the canonical neighborhood assumptions in this way.

Consider Ricci flows $M_k \times I_k$ that smoothly converge to a limit flow $M_\infty \times I_\infty$ and let $(x_k, t_k) \in M_k \times I_k$ be a sequence that converges to $(x_\infty, t_\infty) \in M_\infty \times I_\infty$ with $S(x_\infty, t_\infty) > 0$. Assume that the (x_k, t_k) are centers of strong ε-necks in the sense above. We want to make clear that then (x_∞, t_∞) is the center of a strong 2ε-neck if ε is smaller than some universal constant. This fact is certainly true if we use the conventional definition of strong 2ε-necks. Denote by $U_k \times [t_k - \tau_k, t_k]$ strong ε-necks with center (x_k, t_k). For small ε we have $B_k := B_{0.9\varepsilon^{-1}S^{-1/2}(x_k,t_k)}(x_k, t_k) \subset U_k$ for all k. From the uniform estimates on the curvature derivatives and the noncollapsedness we find that the sequence of pointed Ricci flows $(B_k \times [t_k - \tau_k, t_k], (x_k, t_k))$ smoothly subconverges to some solution $(B_\infty \times [t_\infty - \tau_\infty, t_\infty], (x_\infty, t_\infty))$ whose final time slice can be identified with the ball $B_{0.9\varepsilon^{-1}S^{-1/2}(x_\infty,t_\infty)}(x_\infty, t_\infty) \subset M_\infty(t_\infty)$ and whose metric coincides with the metric of $M_\infty \times I_\infty$ on $B_\infty \times [t_\infty - \tau_\infty, t_\infty] \cap M_\infty \times I_\infty$ via this identification. Obviously, $B_\infty \times [t_\infty - \tau_\infty, t_\infty]$ contains a strong 2ε-neck $U_\infty \times [t_\infty - \tau', t_\infty]$ with center (x_∞, t_∞).

It is easy to see that $U_\infty \times [t_\infty - \tau', t_\infty]$ has nonnegative sectional curvature if the $U_k \times [t_k - \tau_k, t_k]$ have φ_k-positive curvature with $\varphi_k \to 0$. In a similar way we can also show that if M_k is a sequence of Riemannian manifold that Gromov-Hausdorff converges to some metric space X_∞ and $x_k \to x_\infty$ are centers of strong ε-necks then the Gromov-Hausdorff convergence is smooth in x_∞ and x_∞ is the center of a strong 2ε-neck.

6.2 Bounded curvature at bounded distances

Lemma 6.2.1. *Let $\eta, r > 0$, M be a Riemannian manifold and $\sigma : [0, l] \to M$ a curve parameterized by arclength that connects two points $x, y \in M$. Assume that $\|\nabla S^{-1/2}(z)\| \leq \eta$ at any point $z \in \sigma$ for which $S^{-1/2}(z) < r$ (we set $S^{-1/2}(z) = \infty$ if $S(z) \leq 0$).*
Now if $S^{-1/2}(x)$ or $S^{-1/2}(y) < r - \eta l$, then $S^{-1/2} < r$ on σ and $|S^{-1/2}(x) - S^{-1/2}(y)| \leq \eta l$.

Proof. The second assertion is obvious. For the first assume that $S^{-1/2}(x) < r - \eta l$. Let $s \in [0, l]$ be maximal with the property that $S^{-1/2}(\sigma(s)) \leq r$. Then $S^{-1/2}(\sigma(s)) < r - \eta(l - s)$. So we must have $s = l$ and the assertion follows. $\square$

Analogously we prove

Lemma 6.2.2. *Let $\eta, r > 0$, $M \times I$ be a Ricci flow, $x \in M$ and $[t_1, t_2] \subset I$. Assume that $|\partial_t S^{-1}(x, t)| \leq \eta^2$ at any time $t \in [t_1, t_2]$ for which $(S^+)^{-1}(x, t) < r^2$ (where $S^+ := \max\{0, S\}$).*
Now if $(S^+)^{-1}(x, t_1)$ or $(S^+)^{-1}(x, t_2) < r - \eta^2 r^2$, then $(S^+)^{-1}(x, \cdot) < r$ on $[t_1, t_2]$ and $|(S^+)^{-1}(x, t_1) - (S^+)^{-1}(x, t_2)| < \eta^2 r^2$.

Lemma 6.2.3. *There is an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ the following holds:*
Let $\kappa, \rho, r > 0$ and M be a 3 dimensional Riemannian manifold of nonnegative sectional curvature that is κ-noncollapsed on scales $< \rho$. Furthermore, let $\gamma : [0, s_0) \to M$ (where $s_0 \in \mathbb{R}_+ \cup \{\infty\}$) be a minimizing geodesic such that $S(\gamma(s))$ does not stay bounded for $s \to s_0$ and assume that all points on γ are centers of strong ε-necks.
Then $s_0 = \infty$.

Proof. Observe first that for ε_0 sufficiently small we may assume that $S > 0$ and $\|\nabla S^{-1/2}\| < 1$ on all ε-necks.

Assume that $s_0 < \infty$. Integrating the estimate from the last paragraph, we conclude that $S^{-1/2}(\gamma(s)) \leq s_0 - s$ for all $s \in [0, s_0)$ and thus $S(\gamma(s)) \to \infty$ for $s \to s_0$.

From Proposition 5.4.7 we get that γ lies inside some open set $N \subset M$ that can be diffeomorphically identified with $S^2 \times (0, 1)$ and which is covered by final time slices of strong ε-necks. In this identification we may assume that $\lim_{s \to s_0} \mathrm{pr}_2 \circ \gamma(s) = 1$. Consider the path metric induced by the Riemannian metric on N. Let $p := \lim_{s \to s_0} \gamma(s) \in \overline{N}$ where $\overline{N}$ denotes the completion of N and set $N' := N \cup \{p\}$. Since the widths of the ε-necks that cover N approach 0 in the direction to p, any sequence $(x_k, s_k) \in S^2 \times (0, 1) = N$ with $s_k \to 1$ converges to p. This shows that for small d_0 the closed d-balls $\overline{B}_d(p) \subset N'$ are complete for $d < d_0$.

Obviously, the metric on N' is still a length metric. Furthermore, observe that any curve σ that connects two points $x, y \in N$ via p has to pass a whole ε-neck. So for ε_0 sufficiently small, σ can be replaced by a shorter curve joining x and y. We conclude that for sufficiently small d_0 any two points $x, y \in B_{d_0}(p) \setminus \{p\}$ can be joined by a minimizing (Riemannian) geodesic in N' and any such geodesic doesn't hit p. Moreover, for any $x \in B_{d_0}(p)$ there is a minimizing geodesic between x and p in N' realized as the limit of minimizing geodesics between x and y for $y \to p$.

The preceding conclusion enables us to deduce the Bishop-Gromov Theorem 1.4.2 in the case $\mathrm{Ric} \geq 0$ for concentric balls contained in $B_{d_0}(p)$ whose center is not p. So by the proof of Proposition 3.1.2, for any d the d-balls $B_d^{\lambda N'}(p)$ in the rescalings $\lambda N'$ of N' are uniformly totally bounded for $\lambda \to \infty$.

From Toponogov's Theorem 1.3.1 and the remarks below it we find[1] that for sufficiently small d_0 all quadruples of distinct points $x_0, x_1, x_2, x_3 \in B_{d_0}(p) \setminus \{p\}$ satisfy the inequality

[1]There is a little subtlety since $B_{d_0}(p) \setminus \{p\}$ is not complete. However by the preceding results, all minimizing geodesics that are used in the proof of Toponogov's Theorem do not hit p.

in (C). By continuity this inequality is also fulfilled for all quadruples of distinct points $x_0, x_1, x_2, x_3 \in B_{d_0}(p)$. So N' is locally Alexandrov in p.

Proposition 3.5.1 gives us that we have Gromov-Hausdorff convergence

$$(\lambda N', p) \xrightarrow[\lambda \to \infty]{} (C, p_\infty) \tag{6.1}$$

where (C, p_∞) is the tangential cone of N' in p. We will show that this convergence is actually smooth on C_0.

We first deduce a lower estimate for S on $B_{d_0}(p)$. Let $x \in B_{d_0}(p) \setminus \{p\}$ and $s > 0$ small. By the inequality in the first paragraph we have $S^{-1/2}(x) \leq S^{-1/2}(\gamma(s)) + \operatorname{dist}(x, \gamma(s))$. In the limit $s \to s_0$ this implies

$$S \geq \frac{1}{\operatorname{dist}^2(p, \cdot)} \qquad \text{on} \qquad B_{d_0}(p). \tag{6.2}$$

Now we want to estimate S from above. Choose a minimizing geodesic $\gamma' : [0, l] \to N \cup \{p\}$ with $\gamma'(l) = p$ such that the comparison angle $\alpha := \tilde{\sphericalangle}\gamma(0)p\gamma'(0) > 0$. Suppose that $d_0 < l$. Let $z \in B_{d_0}(p) \setminus \{p\}$. We know that z lies in some ε-neck $U \subset N$. Let λ be the scaling factor of the corresponding ε-homothety. For sufficiently small ε_0 we can assume that $\frac{1}{2}\lambda^{-2} < S(z) < 2\lambda^{-2}$, that $\operatorname{diam} U < 3\varepsilon^{-1}\lambda$ and that the cross-sections of U have width $< 3\pi\lambda$. The geodesics γ and γ' intersect a cross-section S of U in some points $q := \gamma(s)$ and $q' := \gamma'(s')$. From property (B) of Definition 1.3.2 (the monotonicity of the comparison angle) applied to γ and γ' we get $\widetilde{\alpha} := \tilde{\sphericalangle} q'pq \geq \alpha$. Consider the triangle $\triangle qpq'$ and its Euclidean comparison triangle $\triangle \widetilde{q}\widetilde{p}\widetilde{q}'$. Let $\widetilde{\gamma}$ be the line throught $\widetilde{p}$ and $\widetilde{q}$. Then $\operatorname{dist}(\widetilde{p}, \widetilde{q}) \sin \widetilde{\alpha} = \operatorname{dist}(\widetilde{q}', \widetilde{\gamma}) \leq \operatorname{dist}(\widetilde{q}', \widetilde{p})$, hence

$$(s_0 - s) \sin \alpha \leq \operatorname{dist}(q, q') < 3\pi\lambda.$$

By the triangle inequality $\operatorname{dist}(p, z) - 3\varepsilon^{-1}\lambda < s_0 - s$ and thus

$$S(z) < 2\lambda^{-2} < \frac{18(\pi \sin^{-1}\alpha + \varepsilon^{-1})^2}{\operatorname{dist}^2(p, z)}.$$

The estimate implies together with the fact that the points in $B_{d_0}(p)$ lie in strong ε-necks that the convergence (6.1) is smooth on C_0. From (6.2) we deduce that C_0 is nowhere flat. Moreover, any point $z \in C_0$ is the center of a strong 2ε-neck of nonnegative sectional curvature. This however contradicts Lemma 2.10.1. $\square$

Proposition 6.2.4 (Bounded curvature at bounded distances). *For ε_0 sufficiently small we have:*
Let $\kappa, \rho, r > 0$, $E, D < \infty$ and $\varepsilon < \varepsilon_0$. Then there are constants $C(D, \kappa, \rho, r, E) < \infty$ and $\varphi(D, \kappa, \rho, r, E) > 0$ such that the following statement holds:
Let (M, g) be a 3 dimensional Riemannian manifold and $p \in M$ such that

- *the curvature is φ-positive,*
- *M is κ-noncollapsed on scales $< \rho$,*
- *the balls $B_{D-\delta}(p)$ are relatively compact in M for any $\delta > 0$,*
- *the canonical neighborhood assumptions $\mathrm{CNA}(r, \varepsilon, E)$ hold for all points in $B_D(p)$ where we interpret strong ε-necks in the sense above and require that those also have φ-positive curvature*

Now if $S(p) \leq 1$, then $S \leq C$ on $B_D(p)$.

Proof. Fix some constants $\kappa, \rho, \varepsilon = \varepsilon_0$ and E. If the proposition is true for $D \geq 0$ with constants ρ and $C > \frac{1}{r^2}$ then it also holds for $D + \frac{r}{2\eta\sqrt{C}}$ and C replaced by $4C$ by Lemma 6.2.1. This shows that the set

$$I_{\kappa,\rho,r,E} := \{D \in \mathbb{R}^+ \ : \ \text{the proposition is true for } D\}$$

is nonempty and open. Assume that $I_{\kappa,\rho,r,E} \neq [0,\infty)$ and let $D := \sup I_{\kappa,\rho,r,E} < \infty$. Choose a sequence $\varphi_k \to 0$. We can find Riemannian manifolds (M_k, g_k) with $p_k \in M_k$ that satisfy the assumptions of the proposition (this includes that the M_k and the strong ε-necks are φ_k-pinched towards positive curvature) but for which we can find points $q_k \in B_D^{M_k}(p_k)$ with $S(q_k) \to \infty$. So $l_k := \operatorname{dist}(p_k, q_k) \to D$.

For any $\delta > 0$ we have uniform bounds for S on $B_{D-\delta}(p_k)$. Since the M_k have φ_k-positive curvature, these bounds give bounds for $\|R\|$. Together with the κ-noncollapsedness on scales $< \rho$ this implies that after passing to a subsequence we have (metric) Gromov-Hausdorff convergence

$$\left(\overline{B}_D^{M_k}(p_k), p_k\right) \xrightarrow[k\to\infty]{} (X_\infty, p_\infty)\,. \tag{6.3}$$

Here $\overline{B}_D^{M_k}(p_k)$ denotes the metric completion of $B_D^{M_k}(p_k)$ which is compact. (There is a little difficulty in showing that the $\overline{B}_D^{M_k}(p_k)$ are uniformly totally bounded: For any $\delta > 0$ the κ-noncollapsedness gives us a uniform lower bound on the volume of δ-balls contained in $B_D(p_\infty)$. Furthermore, from Bishop-Gromov's Theorem 1.4.2 we get uniform upper bounds for the volume of the balls $B_{D-\delta}^{M_k}(p_k)$. As in Proposition 3.1.2 both bounds give upper bounds for the cardinality of minimal 2δ-nets of $B_{D-\delta}^{M_k}(p_k)$. Such nets are 3δ-nets for $B_D^{M_k}(p_k)$.)

Choose minimizing geodesics $\gamma_k : [0, l_k] \to M_k$ parameterized by arclength between p_k and q_k. After passing to a subsequence we may assume that we have convergence $\gamma_k \to \gamma_\infty$ and $q_k \to q_\infty$ such that $\gamma_\infty : [0, D] \to X_\infty$ is a minimizing geodesic and $\gamma_\infty(D) = q_\infty$. Since $S^{-1/2}(q_k) \to 0$, Lemma 6.2.1 gives us that

$$\limsup_{k\to\infty} S^{-1/2}(\gamma_k(D-s)) \leq \eta s \tag{6.4}$$

for $0 < s < \frac{r}{\eta}$. Together with Lemma 5.4.8 this implies that for small s the points $\gamma_k(D-s)$ are eventually all centers of strong ε-necks. As we have uniform bounds for the curvature on the balls $B_{D-\delta}^{M_k}(p_k)$, there is a neighborhood U around $\gamma_\infty|_{(D-s,D)}$ for small s on which the convergence (6.3) is smooth and the points on $\gamma_\infty|_{(D-s,D)}$ are centers of strong 2ε-necks in U. Since $\varphi_k \to 0$, the sectional curvature on U is nonnegative and the strong 2ε-necks also have nonnegative sectional curvature. From (6.4) we conclude that $S(\gamma_\infty(D-s)) \to \infty$ for $s \to 0$ contradicting Lemma 6.2.3. □

6.3 The Canonical Neighborhood Theorem

Proposition 6.3.1. *For $\varepsilon_0 > 0$ sufficiently small, $\kappa, \rho, r > 0$, $E < \infty$ and $\varepsilon < \varepsilon_0$ the following statement holds:*
Let $M_k \times [-T_k, 0]$ be a sequence of 3 dimensional Ricci flows such that

- *the $M_k \times [-T_k, 0]$ have φ_k-positive curvature with $\varphi_k \to 0$,*
- *the time slices of the $M_k \times [-T_k, 0]$ are κ-noncollapsed on scales $< \rho$*
- *the canonical neighborhood assumptions $\mathrm{CNA}(r, \varepsilon, E)$ hold on the $M_k \times [-T_k, 0]$ in the sense explained in section 6.1 (where we assume that the strong ε-necks have φ_k-positive curvature).*
- *there is a sequence $d_k \to \infty$ such that the balls $B_{d_k}^{M_k}(p_k, 0)$ are relatively compact in M_k.*

Let $p_k \in M_k$ with $S(p_k, 0) \leq 1$.
Now if $0 < T_\infty := \limsup_{k\to\infty} T_k \leq \infty$ and $0 \leq T \leq T_\infty$, then the pointed Ricci flows $(M_k \times (-T, 0], (p_k, 0))$ smoothly subconverge to some Ricci flow $(M_\infty \times (-T, 0], (p_\infty, 0))$ such that the time slices are complete and κ-noncollapsed on scales $< \rho$ and the sectional curvature is nonnegative. Moreover, the Riemannian curvature is bounded on $M \times (-T, 0]$.

Observe that we require the time slices of the $M_k \times [-T_k, 0]$ to be κ-noncollapsed on scales $< \rho$ and not the Ricci flows itself. This is a slightly stronger condition. The power of this Proposition lies in the fact that we neither assume the M_k to be complete nor the curvature to satisfy any uniform curvature bound (except at $(p_k, 0)$).

Proof. Proposition 6.2.4 gives us bounds $C(D)$ for $S(\cdot, 0)$ on M_k for large k depending on the distance D to p_k. With the help of Lemma 6.2.2 we can extend these bounds to times $[-\frac{1}{2\eta^2 C(D)}, 0] \cap (-T, 0]$. Since the $M_k \times [-T_k, 0]$ have φ_k-positive curvature, these bounds imply bounds on $\|R\|$ and Shi's estimates give us bounds for the curvature derivatives $\|\nabla^l R\|$. Additionally, using the κ-noncollapsedness on scales $< \rho$, we conclude that after passing to a subsequence we have smooth convergence

$$(M_k(0), p_k) \xrightarrow[k \to \infty]{} (M_\infty(0), p_\infty) \tag{6.5}$$

where $M_\infty(0)$ is a complete Riemannian manifold of nonnegative sectional curvature. Observe that $M_\infty(0)$ has the property that all points $x \in M_\infty(0)$ with $S(x, 0) > \frac{1}{r^2}$ are centers of 2ε-necks or $(2\varepsilon, 2E)$-caps.

Assume that $M_\infty(0)$ doesn't have bounded curvature, i.e. there is a sequence of points $y_l \in M_\infty(0)$ with $S(y_l) \to \infty$. By passing to the universal cover we may assume $M_\infty(0)$ to be simply connected for the moment. From Lemma 5.1.5 we conclude after passing to a subsequence that there is a ray $\sigma : [0, \infty) \to M_\infty(0)$ starting in p_∞ and a sequence $s_l \to \infty$ such that we have $\mathrm{dist}_0(y_l, \sigma(s_l)) \to \infty$ and for the comparison angles $\widetilde{\sphericalangle} p_\infty y_l \sigma(s_l) \to \pi$. So by Lemma 5.4.8 eventually all points y_l are centers of 2ε-necks U_l that separate $M_\infty(0)$ into two parts. Furthermore, Lemma 5.1.5 asserts that $\sigma|_{[s_l, \infty)}$ doesn't hit $B_{\frac{1}{2}\mathrm{dist}_0(p_\infty, y_l)}(y_l, 0)$. So the part of $M_\infty(0) \setminus U_l$ that doesn't contain p_∞ is noncompact. Now we can apply the reasoning in the proof of Corollary 5.1.9 to get a contradiction. Thus $S < Q$ on $M_\infty(0)$ for some constant Q.

We have just shown that the theorem is true for $T = 0$ (if we set $(0, 0] = \{0\}$). Assume that it is true for $T < T_\infty$ with the bound Q, i.e. we can extend the smooth convergence (6.5) for a subsequence to a convergence

$$(M_k \times (-T, 0], p_k) \xrightarrow[k \to \infty]{} (M_\infty \times (-T, 0], p_\infty) \tag{6.6}$$

of Ricci flows such that $S < Q$ on $M_\infty \times (-T, 0]$. Using Lemma 6.2.2 we can extend this convergence for a subsequence to a convergence on $(-T', 0]$ where $T' = \min(T + \frac{Q}{\eta^2}, T_\infty)$. So the set I of times T for which we can extend the convergence (6.5) to Ricci flows on the time interval $(-T, 0]$ to get a Ricci flow $M_\infty \times (-T, 0]$ of bounded curvature, is open in $[0, T_\infty]$.

Assume that $I \neq [0, T_\infty]$ and set $T := \sup I$. Consider first the case $T_\infty < \infty$. After passing to a subsequence we can extend the convergence (6.5) to a convergence (6.6). Now the limit $M_\infty \times (-T, 0]$ must have complete time slices, nonnegative sectional curvature and *unbounded* curvature but bounded curvature on $M_\infty \times [-T', 0]$ for any $T' < T$. This fact enables us to apply the Harnack inequality for the Ricci flow, Theorem 2.8.1. We get that $\frac{\mathrm{d}}{\mathrm{d}t}((t + T')S(\cdot, t)) \geq 0$ for any $T' < T$ and hence $(t + T)S(\cdot, t)$ is pointwise nondecreasing. Let Q be a bound for S on $M_\infty(0)$. We conclude

$$S(\cdot, t) \leq Q \frac{T}{t + T} \qquad \text{on} \qquad M_\infty \times (-T, 0].$$

By the distance distortion estimates of Theorem 2.3.2 there is a constant C' such that for $C := C' \int_{-T}^0 \sqrt{\frac{QT}{t+T}} \mathrm{d}t = 2C' \sqrt{Q} T$ we have

$$\mathrm{dist}_0(x, y) \leq \mathrm{dist}_t(x, y) \leq \mathrm{dist}_0(x, y) + C \tag{6.7}$$

for any $x, y \in M_\infty$ and $t \in (-T, 0]$.

If M_∞ is compact, this result implies that the diameter $\operatorname{diam}_t M_\infty$ stays bounded in t. Since $\min_{M_\infty} S(\cdot, t)$ is nondecreasing, we can find a point $x_t \in M_\infty$ for any time $t \in (-T, 0]$ such that $S(x_t, t) \leq \min_{M_\infty} S(\cdot, 0)$. Apply Proposition 6.2.4 at x_t to show that we have bounded curvature on $M_\infty \times (-T, 0]$ (Observe that $M_\infty \times (-T, 0]$ satisfies the canonical neighborhood assumptions $\mathrm{CNA}(\frac{1}{2}r, 2\varepsilon, 2E)$).

Assume now that M_∞ is noncompact and without loss of generality simply connected. First we claim that there is a sequence $(y_l, t_l) \in M_\infty \times (-T, 0]$ such that $S(y_l, t_l) \to \infty$ and $\operatorname{dist}_0(p_\infty, y_l) \to \infty$. If not, the curvature would be bounded on $(M \setminus B_D(p_\infty, 0)) \times (-T, 0]$ for some D and we could apply (6.7) and Proposition 6.2.4 with basepoint in this set to derive a contradiction. For large l the points (x_l, t_l) are centers of 2ε-necks or 2ε-caps. Lemma 5.1.5 gives us after passing to a subsequence a time 0 ray $\sigma : [0, \infty) \to M_\infty$ starting in p_∞ and a sequence s_l such that

$$\operatorname{dist}_0(y_l, \sigma(s_l)) \to \infty \qquad \text{and} \qquad \widetilde{\sphericalangle}_0 p_\infty y_l \sigma(s_l) \to \pi.$$

From (6.7) we conclude that we have uniform convergence

$$\operatorname{dist}_t(y_l, \sigma(s_l)) \to \infty \qquad \text{and} \qquad \widetilde{\sphericalangle}_t p_\infty y_l \sigma(s_l) \to \pi.$$

for all $t \in (-T, 0]$. So by Lemma 5.4.8 eventually all points (y_l, t_l) are centers of 2ε-necks U_l that separate M_∞ such that p_∞ and $\sigma(s_l)$ lie in different parts of $M_\infty \setminus U_l$. We conclude that the part in which $\sigma(s_l)$ lies, is noncompact since Lemma 5.1.5 asserts that $\sigma|_{[s_l, \infty)}$ does not hit $B_{\frac{1}{2}\operatorname{dist}_0(p_\infty, y_l)}(y_l, 0)$. Since $\operatorname{diam}_0 U_l \leq \operatorname{diam}_{t_l} U_l \to 0$ we can again apply the reasoning of the proof of Corollary 5.1.9 to get a contradiction.

In the case $T_\infty = \infty$ we get the required curvature bound directly from the integrated Harnack inequality for the Ricci flow (see Theorem 2.8.2). Observe that the factor "$\frac{t_1}{t_2}$" can be assumed to be equal to 1 since the flow is ancient. ☐

In the following we switch back to the conventional definition of strong ε-necks.

Theorem 6.3.2 (Canonical Neighborhood Theorem). *Let $\kappa, \rho, \tau, \varphi, \varepsilon > 0$. Then there are constants $r(\kappa, \rho, \tau, \varphi, \varepsilon) > 0$ and $E'(\varepsilon) < \infty$ with the following property:*
Let $M \times [-\tau, 0]$ be a 3 dimensional, orientable Ricci flow

- *with complete time slices,*
- *bounded curvature,*
- *φ-positive curvature (this implies in particular $\tau < \varphi^{-1}$)*
- *that is κ-noncollapsed on scales $< \rho$.*

Then every point at time 0 satisfies the canonical neighborhood assumptions $\mathrm{CNA}(r, \varepsilon, E')$ or M is a higher spherical space form.

Proof. Fix constants $\kappa, r, \tau, \varphi, \varepsilon$, assume $\varepsilon < \varepsilon_0$ and set $E'(\varepsilon) := 2E_{\kappa_0}(\frac{\varepsilon}{2})$, where E denotes the constant obtained in Theorem 5.4.11 and κ_0 the constant from Theorem 5.4.13.

At first choose some $r > 0$ and consider a counterexample, i.e. a Ricci flow $M \times [-\tau, 0]$ that satisfies the assumptions of the theorem but violates the canonical neighborhood assumptions $\mathrm{CNA}(r, \varepsilon, E')$ at some point $(x, 0) \in M \times [-\tau, 0]$. By a point-picking argument (compare also Lemma 5.1.4) we argue that we may assume the canonical neighborhood assumptions $\mathrm{CNA}(\frac{1}{2}r, \varepsilon, E')$ to hold on $M \times [-\frac{\tau}{4}, 0]$: Assume that there is some point $(x_1, t_1) \in M \times [-\frac{\tau}{4}, 0]$ violating $\mathrm{CNA}(\frac{1}{2}r, \varepsilon, E')$. Shift the solution $M \times [-\frac{\tau}{4}, 0]$ by $-t_1$, parabolically rescale by 2 and restrict it to the time interval $[-\tau, 0]$. The resulting flow $M' \times [-\tau, 0]$ still satisfies the assumptions of the theorem (including the φ-positive curvature condition since $\tau < \varphi^{-1}$). Observe that the point $(x', 0) \in M' \times [-\tau, 0]$ corresponding to the point (x_1, t_1) violates $\mathrm{CNA}(r, \varepsilon, E')$. So we have constructed another counterexample for r. Repeat this step as long as not all points on $M' \times [-\frac{\tau}{4}, 0]$ satisfy $\mathrm{CNA}(2r, \varepsilon, E')$.

It is clear that the process has to stop after a finite number of steps since otherwise we would produce a sequence $(x_l, t_l) \in M \times [-\frac{\tau}{3}, 0]$ with $S(x_l, t_l) \to \infty$.

Assume that the conclusion of the theorem was wrong. Choose a sequence $r_k \to 0$. Then we find a sequence of counterexamples $M_k \times [-\tau, 0]$ such that for some point $(x_k, 0) \in M_k \times [-\tau, 0]$ the canonical neighborhood assumptions $\mathrm{CNA}(r_k, \varepsilon, E')$ are violated but $\mathrm{CNA}(\frac{1}{2}r_k, \varepsilon, E')$ is satisfied on $M_k \times [-\frac{\tau}{4}, 0]$. Assume furthermore that the M_k are not higher spherical space forms. So $Q_k := S(x_k, 0) \geq r_k^{-2} \to \infty$. Parabolically rescale the $M_k \times [-\tau, 0]$ by the factor $\lambda_k := Q_k^{1/2}$. This produces Ricci flows $M'_k \times [-\lambda_k^2 \tau, 0]$ that have $\frac{1}{\lambda_k^2}\rho$-positive curvature, are κ-noncollapsed on scales $< \lambda_k \rho$ and that satisfy $S(x'_k, 0) = 1$ (here $x'_k \in M'_k$ denotes the point corresponding to $x_k \in M_k$). Furthermore, $\mathrm{CNA}(1, \varepsilon, E')$ is violated at the point $(x'_k, 0)$, but $\mathrm{CNA}(2, \varepsilon, E')$ holds on $M'_k \times [-\frac{1}{4}\lambda_k^2 \tau, 0]$. Using Lemma 6.2.2 and $\mathrm{CNA}(2, \varepsilon, E')$, we conclude that there is a $\kappa' > 0$ such that the *time slices* of $M'_k \times [-\frac{1}{8}\lambda_k^2 \tau, 0]$ are κ'-noncollapsed on scales $< \rho$. Applying Proposition 6.3.1, we find that we have smooth convergence

$$\left(M'_k \times [-\lambda_k^2 \tau, 0], (x'_k, 0)\right) \xrightarrow[k \to \infty]{} (M_\infty \times (-\infty, 0], (x_\infty, 0))$$

where $M_\infty \times (-\infty, 0]$ is an orientable, ancient solution of bounded nonnegative sectional curvature that is κ'-noncollapsed on all scales and which is not a higher spherical space form. Moreover $S(x_\infty, 0) = 1$, hence it is a κ'-solution.

But Theorems 5.4.11 and 5.4.13 give us now that $M_\infty \times (-\infty, 0]$ is κ_0-noncollapsed and $(x_\infty, 0)$ is the center of a strong $\frac{\varepsilon}{2}$-neck or an $(\frac{\varepsilon}{2}, E_{\kappa_0}(\frac{\varepsilon}{2}))$-cap. So for large k the point $(x'_k, 0)$ must be at least the center of a strong ε-neck or an (ε, E')-cap and the $(\kappa_0 - \varepsilon)$-noncollapsedness assumption must apply on sclaes < 1 at $(x'_k, 0)$. A contradiction. $\square$

Chapter 7

Construction of a Ricci flow with surgery

7.1 Ricci flows with surgery

We will first give a definition for Ricci flows with surgery in general.

Consider a time interval $I \subset \mathbb{R}$. Let $T_1 < T_2 < \ldots < T_{k-1}$ be times in I and divide I into the intervals

$$I^1 := I \cap (-\infty, T_1),\ I^2 := I \cap [T_1, T_2),\ I^3 := I \cap [T_2, T_3),\ \ldots,\ I^k := I \cap [T_{k-1}, \infty).$$

Let $(M^1 \times I^1, g^1), \ldots, (M^k \times I^k, g^k)$ be Ricci flows and let $\Omega^i \subset M^i$ be open sets on which the metric g^i_t smoothly converges to some Riemannian metric $g^i_{T_i}$ on Ω^i for $t \nearrow T_i$. Let furthermore

$$U^i_- \subset \Omega^i \qquad \text{and} \qquad U^i_+ \subset M^{i+1} \qquad \text{for } i = 1, \ldots, k-1$$

be open subsets such that there are isometries

$$\Phi^i : (U^i_-, g^i_{T_i}) \longrightarrow (U^i_+, g^{i+1}_{T_i}), \qquad (\Phi^i)^* g^{i+1}_{T_i}|_{U^i_+} = g^i_{T_i}|_{U^i_-}.$$

Then we call $\mathcal{M} := ((M^\cdot \times I^\cdot, g^\cdot)), (U^\cdot_\pm), (\Phi^\cdot))$ a *Ricci flow with surgery* and $T_1, \ldots, T_{k-1}$ the *surgery times.* Note that if we glue the Ricci flows on $U^i_- \times I^i$ and $U^i_+ \times I^{i+1}$ together via the isometry Φ^i, we obtain a (smooth) Ricci flow on $U^i_- \times (I^i \cup I^{i+1})$.

If $t \in I^i$, then $\mathcal{M}(t) := (M^i \times \{t\}, g^i_t)$ is called the *time t slice* of $\mathcal{M}$. For $t = T_i$ we define the *(presurgery) time T_i^- slice* to be $\mathcal{M}(T_i^-) := (\Omega^i \times \{T_i\}, g^i_{T_i})$ and the *(postsurgery) time T_i^+ slice* to be $\mathcal{M}(T_i^+) := (M^{i+1} \times \{T_i\}, g^{i+1}_{T_i})$. The points $\Omega^i \times \{T_i\} \setminus U^i_- \times \{T_i\}$ are called *presurgery points* and the points $M^{i+1} \times \{T_i\} \setminus U^i_+$ *postsurgery points.* In the course of this chapter we will sometimes need to exclude the presurgery points. For this we call a point that is not a presurgery point a *non-presurgery point.*

For $(x, t) \in \mathcal{M}$ consider a spatially constant line in $\mathcal{M}$ that starts in (x, t) and goes forwards or backwards in time for some time $\Delta t \in \mathbb{R}$ and that doesn't hit any (pre- or post-)surgery points. When crossing surgery times, we can continue the line via the isometries Φ^i. We denote the endpoint of this line by $(x, t + \Delta t) \in \mathcal{M}$. Observe that this point is only defined if there are no surgery points between (x, t) and $(x, t + \Delta t)$. We say that a point $(x, t) \in \mathcal{M}$ *survives until time* $t + \Delta t$ if the point $(x, t + \Delta t) \in \mathcal{M}$ is well-defined.

Using this notion we can define parabolic neighborhoods in $\mathcal{M}$. Let $(x, t) \in \mathcal{M}$ (pre- or postsurgery points are allowed, in this case we have to replace t by either t^- or t^+), $d \geq 0$ and $\Delta t \in \mathbb{R}$. Consider the ball $B := B_d(x, t) := B_d^{\mathcal{M}(t)}(x, t)$. If t is a surgery time, we have to distinguish between $B_d(x, t^-)$ and $B_d(x, t^+)$. For each $(x', t) \in B_d(x, t)$ consider the union $I^{\Delta t}_{x', t}$ of all points $(x', t + t')$ for $t' \in [0, \Delta t]$ resp. $[\Delta t, 0]$. We say that $I^{\Delta t}_{x',t}$ is *non-singular* if $(x', t + \Delta t) \in I^{\Delta t}_{x',t}$. Now define the *parabolic neighborhood* $P(x, t, d, \Delta t) := \bigcup_{x' \in B} I^{\Delta t}_{x',t}$. We call $P(x, t, d, \Delta t)$ *non-singular* if all the $I^{\Delta t}_{x',t}$ are non-singular.

We say that a smooth map $\Psi : N \times J \to \mathcal{M}$ from a Ricci flow into a Ricci flow with surgery *preserves time slices* if for every $t \in J$ there is some $t' \in I$ such that $\Psi(N \times \{t\}) \subset \mathcal{M}(t'), \mathcal{M}(t'^{-})$ or $\mathcal{M}(t'^{+})$. It is called *time-equivariant* if there is a Δt such that for every $x \in N$ there is an x' such that we have $\Psi(x, t) = (x', t + \Delta t)$ for every $t \in J$ and if this definition makes sense (i.e. if Ψ doesn't hit any surgery points in its interior). Ψ is called an *ε-isometry* for some $\varepsilon > 0$ if it preserves time slices, is time equivariant and $\Psi|_{N(t)} : N(t) \to \mathcal{M}(t')$ is an ε-isometry for all $t \in J$ and the corresponding $t' \in I$. Analogously we define *ε-homotheties* into Ricci flows with surgery.

Analogous to the Definition 4.2.1 of κ-noncollapsedness we say that $\mathcal{M}$ is *κ-noncollapsed on scales* $< \rho$ in $(x, t) \in \mathcal{M}$ if $\operatorname{vol}_t B_r(x, t) \geq \kappa r^n$ for all $0 < r < \rho$ for which
(i) the ball $B_r(x, t)$ is relatively compact in $\mathcal{M}(t)$
(ii) the parabolic neighborhood $P(x, t, r, -r^2)$ is nonsingular and
(iii) $\|R\| < \frac{1}{r^2}$ on $P(x, t, r, -r^2)$.

We say that $U \times J \subset \mathcal{M}$ is a *strong ε-neck* if $\{u\} \times J$ is is nonsingular and does not contain surgery points for all $u \in U$ and $U \times J$ is a conventional strong ε-neck with respect to the restricted metric. Observe that strong ε-necks may cross surgery times as long as they don't cross surgery points. It is clear how to transfer notions like φ-positive curvature, ε-necks, (ε, E)-caps or the canonical neighborhood assumptions to the case of Ricci flows with surgery.

7.2 The surgery process

Let $(M, g) = (M^1, g^1_0)$ be a (not necessarily connected) 3 dimensional Riemannian manifold that

(i) is compact, orientable and has no component which is a higher spherical space form,

(ii) has scalar curvature $S < 1$ everywhere,

(iii) is 1-noncollapsed on scales < 1 and

(iv) satisfies the 1-positive curvature condition at time 0.

We will then say that M has *normalized geometry.* If M is the first time slice of a Ricci flow with surgery $\mathcal{M}$ defined on some time interval $[0, T)$, we say that $\mathcal{M}$ has *normalized initial conditions.* Observe that every Riemannian manifold satisfying (i) can be rescaled to have normalized geometry.

We will now describe how to construct the Ricci flow with surgery starting with a Riemannian manifold (M^1, g^1_0) of normalized geometry. By Hamilton's existence results (see Theorems 2.7.1 and 2.7.2) there is a Ricci flow $(M^1 \times [0, T_1), (g^0_\cdot))$ with start metric g^1_0 on a maximal time interval $[0, T_1)$ and $\max_M \|R\|(\cdot, t)$ is unbounded for $t \nearrow T_1$. By property (ii) of the normalized initial conditions and Corollary 2.5.5 we know that $T_1 > \frac{1}{2}$.

Now, consider a more general situation in which surgeries could have already been performed. Let $\mathcal{M}$ be a Ricci flow with surgery and normalized initial conditions defined on some time interval $[0, T)$ such that the curvature is everywhere 1-positive and let $M^k \times [T_{k-1}, T_k)$ with $T_k = T$ be the final Ricci flow in $\mathcal{M}$. Assume that $M \times [T_{k-1}, T_k)$ is defined on a maximal time interval. In order to explain the geometry at the singular time T_k we assume that the canonical neighborhood assumptions $\mathrm{CNA}(r, \varepsilon, E)$ hold at all non-presurgery times for some constants $r > 0$, $\varepsilon < \varepsilon_0$ and $E < \infty$. This is a very crucial assumption. The way the surgeries will be performed strongly depend on these parameters. In the subsequent sections we will prove that the canonical neighborhood assumptions are always satisfied for certain parameters that are *independent* of M and just depend on time in case alle preceding surgeries have been performed in the way described below.

step 1: Determination of the regions on which the metric converges.
Let $\Omega^k \subset M^k$ be the set of points on which $S(\cdot,t)$ stays bounded for $t \nearrow T$. Lemmas 6.2.1 and 6.2.2 imply that $S(\cdot,t)$ is locally bounded on Ω^k, so Ω^k is open. The 1-positive curvature condition implies that $\|R\|(\cdot,t)$ stays locally bounded for $t \nearrow T$ and Shi's estimates give local bounds for the curvature derivatives $\|\nabla^l R\|$ on Ω^k. We conclude that g_t^k smoothly converges to some limit metric g_T^k on Ω^k. It is easy to see that $S(x,T) \to \infty$ for $x \to \partial\Omega^k$. Thus the set

$$\Omega_0^k := \left\{ x \in \Omega \quad : \quad S(x,T) < \frac{4}{r^2} \right\} \subset \Omega^k$$

is relatively compact in Ω^k. From Lemma 6.2.2 we conclude that $S(\cdot,t) \to \infty$ uniformly on $M^k \setminus \Omega^k$ for $t \nearrow T$. Observe that the points on Ω^k satisfy the canonical neighborhood assumptions $\mathrm{CNA}(\frac{1}{2}r, 2\varepsilon, 2E)$.

step 2: Characterization of the topology of $M^k \setminus \Omega_0^k$.
From Proposition 5.4.7 (applied to a time T' slightly smaller than T) we know that $M^k \setminus \Omega_0^k$ is covered by open sets N_i that are diffeomorphic to one of the following manifolds

$$S^2 \times \mathbb{R},\ \mathbb{B}^3,\ \mathbb{R}P^3 \setminus \overline{\mathbb{B}}^3,\ S^3,\ S^2 \times S^1,\ \mathbb{R}P^3 \ \text{ or } \ \mathbb{R}P^3 \# \mathbb{R}P^3 \tag{7.1}$$

which are covered by ε-necks or ε-caps at time T' themselves. It is easy to see (e.g. by a volume argument) that there are only finitely many N_l. We have $\partial N_i \subset \Omega_0^k$ for all i.

step 3: Characterization of the geometry of Ω^k at the ends.
Consider the Riemannian manifold (Ω^k, g_T^k). Applying Proposition 5.4.7 again, yields that $\Omega^k \setminus \Omega_0^k$ is also covered by a finite number of disjoint open sets $N'_i \subset \Omega^k$ diffeomorphic to one of the manifolds in (7.1) which are covered by time T necks or caps themselves. Moreover, by a volume argument, each boundary component of N'_i is either contained in Ω_0^k or $S(\cdot,T)$ diverges on N'_i near this component. Let $\widetilde{\Omega}^k$ be the union of the connected components of Ω^k that are not completely covered by some N'_i. The ends of $\widetilde{\Omega}^k$ are covered by N'_i which are diffeomorphic to $S^2 \times \mathbb{R}$ and that have the property that one boundary component $\partial_1 N'_i$ lies in Ω_0^k and the curvature becomes unbounded near the other component $\partial_2 N'_i$. We call these N'_i *horns*. By the arguments of subsection 5.4.2 we conclude that for every horn $N'_{i'}$ there is a time T strong ε-neck such that one of its cross sectional 2-spheres Σ'_i is parallel to some time T' ε-neck covering an N_i. (Choose a cross section Σ'_i of $N'_{i'}$ close to the boundary component $\partial_1 N'_{i'}$. If T' is close enough to T, we can assume that a time T strong 2ε-neck having Σ'_i as cross section intersects with an ε-neck at time T'.)

step 4: Topological description of the surgery process.
Now choose for each horn N'_i a cross section Σ_i. We will later specify the position of Σ_i more precisely. Observe that Σ'_i and Σ_i bound a set that is diffeomorphic to $S^2 \times [0,1]$. Cut $\widetilde{\Omega}^k$ along the Σ_i, discard the part that does not contain $\partial_1 N'_i$ and glue in 3-balls (i.e. identify the boundary $\partial\overline{\mathbb{B}}^3$ of a closed ball $\overline{\mathbb{B}}^3$ with the produced cutting surface). Perform this cutting and gluing for all horns, discard all higher spherical space forms that arose this way and denote the resulting manifold by M^{k+1}. Obviously, M^{k+1} is compact. It is easy to see that the original manifold M^k can be reconstructed from M^{k+1} by adding some components diffeomorphic to $S^2 \times S^1, \mathbb{R}P^3, \mathbb{R}P^3 \# \mathbb{R}P^3$ or spherical space forms and performing a finite number of connected sums between some of the componentents of M^{k+1} and the additional ones. Note that a surgery that does not seperate the manifold into two components corresponds to the inverse of a connected sum with $S^2 \times S^1$.

step 5: Geometric description of the surgery process.
We now give a description of how to find the Σ_i and how to perform the cutting and gluing *geometrically*. For this we need to find strong δ-necks inside of horns with δ even smaller

than 2ε. Surprisingly, horns have the property that their geometry improves towards their ∂_2-end. We quantify this in the following

Lemma 7.2.1. *For ε_0 sufficiently small and $\varepsilon < \varepsilon_0$ we have:*
For any $\delta > 0$ there is an $0 < h(\delta) < 1$ (depending only on δ) such that if $0 < r < 1$ and we are in the above situation (this includes that the canonical neighborhood assumptions $\mathrm{CNA}(r, \varepsilon, E)$ *apply at all non-presurgery points), then every point x in a horn N_i' with $S(x, T) \geq \frac{1}{h^2 r^2}$ is the center of a strong δ-neck at time T.*

Note that h does not depend on r.

Proof. Assume that $x \in N_i' =: N'$ with $Q := S(x, T) \geq \frac{1}{h^2 r^2}$ for some $0 < h < \frac{1}{2}$. Shift $\mathcal{M}$ in time by $-T$ and parabolically rescale by $Q > \frac{1}{h^2}$ to get $\mathcal{M}'$. Let $(x', 0) \in \mathcal{M}'$ be the point corresponding to $(x, T) \in \mathcal{M}$. We have $S(x', 0) = 1$. For simplicity denote the set in $\mathcal{M}'$ corresponding to N' also by N' and likewise for Ω_0^k. Note that $\mathcal{M}'$ has h^2-positive curvature, the non-presurgery points satisfy $\mathrm{CNA}(r', \varepsilon, E)$ and the points in $\mathcal{M}'(0)$ satisfy $\mathrm{CNA}(\frac{1}{2} r', 2\varepsilon, 2E)$ where $r' = Q^{1/2} r \geq \frac{1}{h} > 2$. In particular this implies that $\mathcal{M}'(0)$ is $\frac{1}{2}\kappa_0$-noncollapsed in $(x', 0)$. Since the boundary component $\partial_1 N'$ lies in Ω_0^k and $S^{-1/2} > \frac{r'}{2}$, Lemma 6.2.1 gives that

$$\mathrm{dist}_0(x', \partial_1 N') \geq \eta^{-1} \left(\frac{r'}{2} - 1 \right) \geq \eta^{-1} \left(\frac{1}{2h} - 1 \right). \tag{7.2}$$

Now assume that the assertion of the Lemma was wrong for some $\delta > 0$. Choose a sequence $h_l \to 0$. We find counterexamples $\mathcal{M}_l$ for h_l together with constants $0 < r_l < 1$, $\varepsilon_l < \varepsilon_0$, E_l, horns N_l', times T_l and points $x_l \in N_l'$ with $S(x_l, T_l) = 1$ that are not centers of strong δ-necks. The preceding paragraph gives us Ricci flows with surgery $\mathcal{M}_l'$ together with numbers $r_l' \geq \frac{1}{h_l}$ such that $\mathcal{M}_l'$ has h_l^2-positive curvature and satisfies $\mathrm{CNA}(r_l', \varepsilon_l, E_l)$ at all non-presurgery points and $\mathrm{CNA}(\frac{1}{2} r_l', 2\varepsilon_l, 2E_l)$ on $\mathcal{M}_l'(0)$. Moreover, there are points $(x_l', 0) \in \mathcal{M}_l'$ with $S(x_l', 0) = 1$ that are not centers of δ-necks and in which $\mathcal{M}_l'(0)$ is $\frac{1}{2}\kappa_0$-noncollapsed. From (7.2) we get that $\mathrm{dist}_0(x', \partial_1 N_l') \to \infty$ for $l \to \infty$. Assume that $D := \liminf_{l \to \infty} \mathrm{dist}_0(x', \partial_2 N_l') < \infty$. Then by the bounded curvature at bounded distances result, Proposition 6.2.4, we have bounds for $S(\cdot, 0)$ on $B_D^{N_l'}(x_l', 0)$ for sufficiently large l. But this would contradict the fact that $S(\cdot, 0)$ diverges on N_l' near $\partial_2 N_l'$. So $\mathrm{dist}_0(x', \partial N_l') \to \infty$.

Observe that again by Proposition 6.2.4 for any $D < \infty$ we have uniform bounds for the curvature on the balls $B_D^{N_l'}(x_l', 0)$. Since N_l' is covered by strong 2ε-necks, these bounds imply bounds on the cuvature derivatives $\nabla^j R$. So we have smooth convergence

$$(N_l' \times (-\tau, 0], (x_l', 0)) \xrightarrow[l \to \infty]{} (M_\infty(0) \times (-\tau, 0], (x_\infty, 0))$$

for a subsequence and a maximal $\tau \geq 0$ (we set $(0, 0] = \{0\}$). $M_\infty \times (-\tau, 0]$ has complete time slices and nonnegative sectional curvature. Moreover, by the proof of Proposition 6.3.1, the curvature on $M_\infty \times (-\tau, 0]$ is uniformly bounded. It is easy to see that $M_\infty(0)$ is covered by 4ε-necks, thus $M_\infty(0) \approx S^2 \times \mathbb{R}$ and the splitting is also a geometric splitting for all times $(-\tau, 0]$.

So there is a bound $C < \infty$ such that for any $D < \infty$ and $\tau' > 0$ we have $\|R\| < C$ on the parabolic neighborhoods $P^{N_l'}(x_l', 0, D, -\tau + \tau')$ for large l. Since the scalar curvature on $M_\infty \times (-\tau, 0]$ must be positive (otherwise M_∞ would be flat at some time by the strong maximum principle, which is impossible in view of the topology), the scalar curvature on $P^{N_l'}(x_l', 0, D, -\tau + \tau')$ can be bounded from below by for large l. Since $r_l' \to \infty$ we conclude by the canonical neighborhood assumptions $\mathrm{CNA}(\frac{1}{2} r_l', 2\varepsilon, 2E)$ that for large l all points on $P^{N_l'}(x_l', 0, D, -\tau + \tau')$ are centers of strong 2ε-necks or $(2\varepsilon, 2E)$-caps. Note that the geometry of this parabolic neighborhood is a almost a product with a line. So we can use

Lemma 5.4.8 to conclude that for large l the points in $P^{N_l'}(x_l', 0, D, -\tau + \tau')$ are centers of strong ε-necks. Thus there is a $\tau'' > 0$ such that we get a bound for the curvature on some larger parabolic neighborhood $P^{N_l'}(x_l', 0, D, -\tau - \tau'')$ that does not depend on D and we may exclude surgery points there. But this contradicts the maximality of τ.

We conclude $\tau = \infty$. Observe again that the scalar curvature on $M_\infty \times (-\infty, 0]$ must be positive. By the canoncial neighborhood assumptions $\mathrm{CNA}(\frac{1}{2}r_l', 2\varepsilon, 2E)$ and the fact that $r_l' \to \infty$ we conclude that $M_\infty \times (-\infty, 0]$ is $\frac{1}{2}\kappa_0$-noncollapsed. Hence it is a $\frac{1}{2}\kappa_0$-solution and therefore isometric to the standard round shrinking cylinder $(S^2 \times \mathbb{R}) \times (-\infty, 0]$. This gives the desired contradiction. $\square$

Before going into the surgery process, we have to describe the geometry that we are going to put on the glued-in balls of step 4. For this we construct a Riemannian manifold $(M_{\mathrm{stan}}, g_{\mathrm{stan}})$ which we will refer to as the *standard cap*. We will later cut along carefully chosen cross sections Σ_i of the horns and glue in a rescaled part of M_{stan}.

In [KL, Sec 72] it is shown that there are constants $B > A > 0$ and a metric on $\mathbb{R}^3$ such that if we identify $\mathbb{R}^3 \setminus \{0\}$ with $S^2 \times (-B, \infty)$ (we assume that 0 lies at the $-B$ end of $S^2 \times (-B, \infty)$), we have

(i) M_{stan} has nonnegative sectional curvature.

(ii) The metric g_{stan} restricted to $S^2 \times (-A, \infty)$ coincides with the standard round metric on $S^2 \times (-A, \infty) \subset S^2 \times \mathbb{R}$.

(iii) The scalar curvature is bounded from below by 1 and attains its maximum $S_0 := S(0)$ at the point $p_{\mathrm{stan}} = 0$ which we call the *tip*.

Consider again the Ricci flow with surgery $\mathcal{M}$ and the manifold $(\widetilde{\Omega}^k, g_T^k)$. For every horn $N_i' \subset \widetilde{\Omega}^k$ we find points $x_i \in V_i$ with $S(x_i, 0) = \max\{\frac{1}{h^2(\delta)r^2}, \frac{1}{\delta^2 r^2}\}$. By the preceding Lemma these points are centers of strong δ-necks. Choose δ-homotheties with factors λ_i

$$\Phi_i : S^2 \times \left(-\tfrac{1}{\delta}, \tfrac{1}{\delta}\right) \to N_i'.$$

Set $h_i := \lambda_i^{-2}\Phi_i^* g_T^k$. We assume $\frac{1}{\delta} \gg B$. In [KL, Lemma 71.20] it is shown that we can find metrics h_i' on $\{p_{\mathrm{stan}}\} \cup S^2 \times (-B, \frac{1}{\delta}) \approx \mathbb{B}^3$ such that

(i) The $\lambda_i^2 h_i'$ satisfy the 1-positive curvature condition for time T.

(ii) h_i' coincides with h_i on $S^2 \times [0, \frac{1}{\delta})$.

(iii) The identity map $(\{p_{\mathrm{stan}} \cup S^2 \times (-B, \frac{1}{\delta}), g_{\mathrm{stan}}) \to (\{p_{\mathrm{stan}} \cup S^2 \times (-B, \frac{1}{\delta}), h_i')$ is a δ'-isometry for all i where $\delta'(\delta)$ depends only on δ and $\delta' \to 0$ for $\delta \to 0$.

(iv) The scalar curvature of h_i' is pointwise not less than the scalar curvature of h_i on $S^2 \times (-B, \frac{1}{\delta})$.

We now cut along the cross sections $\Sigma_i := \Phi_i(S^2 \times \{0\})$, glue in balls and endow the resulting manifold with the push forwards of the metrics $\lambda_i^2 h_i'$ under Φ_i on their images. Outside of the images of the Φ_i the metric remains unchanged. From the arisen components we discard all higher space forms and call the resulting manifold (M^{k+1}, g_T^{k+1}). It is clear that M^{k+1} satisfies the 1-positive curvature condition at time T. We will refer to this procedure in the following as (r, δ)-*cutoff*.

Now consider the Ricci flow $M^{k+1} \times [T_k, T_{k+1})$ with starting metric $g_{T_k}^{k+1}$ on a maximal time interval. Enlarge $\mathcal{M}$ by this solution and call the result $\mathcal{M}'$. By Theorem 2.9.1, $\mathcal{M}'$ still has 1-positive curvature. If $\mathcal{M}'$ satisfies the canoncial neighborhood assumptons $\mathrm{CNA}(r', \varepsilon', E')$ for some r', ε' and E' at all non-presurgery points, we repeat the cutoff process for r', ε' and E'.

Note that after an (r,δ)-cutoff has taken place the manifold has lost a certain amount of volume that depends only on r and δ. Since the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{vol}_t M^k$ of the volume during the Ricci flow $M^k \times [T_{k-1}, T_k)$ is $-\int_{M^k} S \mathrm{d}\mu_t \geq -\frac{3}{1+t}\operatorname{vol}_t M^k$ (by the 1-positive curvature condition), (r,δ)-cutoff times cannot accumulate (for constant r and δ). However, a priori there might exist certain Ricci flows with surgery where we are forced to carry out (r_k,δ_k)-cutoffs at times t_k where $t_k < T < \infty$ for all k, but $\max\{\frac{1}{h^2(\delta_k)r_k^2}, \frac{1}{\delta_k^2 r_k^2}\} \to \infty$. In the following sections we will show that we can control the parameters r, ε, E in the canonical neighborhood assumptions sufficiently well to make this impossible.

7.3 The standard solution

In section 7.4 we will see that shortly after a surgery has taken place, the solution is close to a solution whose start metric is the standard cap $(M_{\text{stan}}, g_{\text{stan}})$. We will refer to these solutions as *standard solutions.*

Definition 7.3.1 (standard solution). *A Ricci flow $(M_{\text{stan}} \times I, g_t)$ (where $I = [0,T)$ or $I = [0,T], T > 0$) with start metric $g_0 = g_{\text{stan}}$ is called a* standard solution *if*

(i) the Riemannian curvature is bounded on compact time intervals

(ii) the sectional curvature is everywhere nonnegative and

(iii) $M_{\text{stan}} \times I$ cannot be extended to a larger time interval $[0,T'] \supsetneqq I$ such that properties (i) and (ii) still hold.

Obviously, every solution $M_{\text{stan}} \times I$ with start metric M_{stan} that satisfies (i) and (ii), can be extended to a standard solution.

We will now derive some useful properties of standard solutions. For this we basically follow an idea of Bernhard Leeb as explained in [KL, Sec 59]. We mention that if we require the standard cap $(M_{\text{stan}}, g_{\text{stan}})$ to be rotationally symmetric in p_{stan}, it is possible to prove that there is only one standard solution (this is a nontrivial statement, see [MT, Thm 12.5] or [KL, Sec 65]). However, the important properties of standard solutions can be proven without using this result.

Theorem 7.3.2. *(a) Every standard solution $M_{\text{stan}} \times I$ is defined on $I = [0,1)$.*

(b) There is a $c > 0$ such that $S(x,t) > \frac{c}{1-t}$ on any standard solution.

(c) Moreover, we have $S \geq 1$ on every standard solution.

(d) For any $t < 1$ there is a $C(t) < \infty$ such that $S < C(t)$ at times $[0,t]$ on every standard solution.

(e) For any $A < \infty$ and any $\delta, \theta > 0$ there is a $B = B(A,\delta,\theta)$ such that for any $x \in M_{stan} \setminus B(p_{\text{stan}}, 0, B)$ on a standard solution $M_{\text{stan}} \times I$ there is a smooth map

$$S^2 \times (-A, A) \xrightarrow{\Phi} M_{\text{stan}} \qquad \text{with} \qquad \Phi(S^2 \times \{0\}) \ni x$$

which is a δ-isometry for times $[0, 1-\theta)$ if we consider the standard round shrinking cylinder metric on $(S^2 \times (-A,A)) \times [0,1)$.

Proof. Let $I_0 = [0,T_0)$ or $[0,T_0]$ ($T_0 \geq 0$) be the maximal interval such that any standard solution is defined for times I_0 and for any compact subinterval $J \subset I_0$ there is a uniform bound for the Riemannian curvature R at times J on all standard solutions $M_{\text{stan}} \times I$. We want to show that $I_0 = [0,1)$.

step 1: For any standard solution $M_{\text{stan}} \times I$ there is a $\kappa > 0$ such that $M_{\text{stan}} \times I$ is κ-noncollapsed at times $I \cap [0,1]$. If $T_0 > 0$, then κ can be chosen independently of $M_{\text{stan}} \times I$.

This follows easily from the No Local Collapsing Theorem 4.2.4 since $(M_{\text{stan}}, g_{\text{stan}})$ is κ'-noncollapsed for some $\kappa' > 0$. Observe that in order to apply Theorem 4.2.4 we need not only curvature control on the first time slice but also for some time t'.

step 2: Let $M_{\text{stan}} \times I$ be a standard solution. Then for any $\varepsilon, \tau > 0$ there is a number $r > 0$ and an $E(\varepsilon)$ such that the canonical neighborhood assumptions $\text{CNA}(r, \varepsilon, E)$ *hold at times $I \cap [\tau, 1]$. Again, if $T_0 > 0$, then the constant r depends only on ε and τ.*
This is an immediate consequence of step 1 and the Canonical Neighborhood Theorem 6.3.2.

step 3: Let $M_{\text{stan}} \times I$ be a standard solution. For any compact $J \subset I$ and any $l \in \mathbb{N}_0$ there is a uniform bound for the curvature derivative $\nabla^l R$ at times J. If $T_0 > 0$, then these bounds are independent of $M_{\text{stan}} \times I$ for all $J \subset I_0$.
Assume $J = [0, T']$. Consider points $x \in M_{\text{stan}}$ far enough from p_{stan} such that $B(x, 0, 1) \subset M_{\text{stan}}(0)$ is isometric to some ball $B(x', 0, 1) \subset S^2 \times \mathbb{R}$ in the standard round cylinder. We may extend the Ricci flow on $B(x, 0, 1)$ backwards to a time interval $(-\tau, 0]$. By the local Shi estimates (see Theorem 2.6.2) we obtain a uniform bound for $\nabla^l R(x, t)$ at times $t \in [0, T']$. Hence outside some compact set $K \subset M_{\text{stan}}$ we can establish the required bound at times $[0, T']$. Now we apply the weak maximum principle on K to the evolution equation of $\|\nabla^l R\|^2$ (see (2.3)). This shows that we can uniformly bound $\nabla^l R$ on $M_{\text{stan}} \times [0, T']$.

step 4: Let $M_{\text{stan}} \times I$ be a standard solution and $x_k \in M_{\text{stan}}$ a sequence such that $\text{dist}_0(p_{\text{stan}}, x_k) \to \infty$*. Then a subsequence of the sequence of pointed Ricci flows $(M_{\text{stan}} \times I, (x_k, 0))$ smoothly converges to the standard round shrinking cylinder $(S^2 \times \mathbb{R}) \times I$. Moreover, $1 \notin I_0$ and any standard solution is at most defined for times $[0, 1]$.*
If $T_0 > 0$ we have: Let $M^k_{\text{stan}} \times I^k$ be a sequence of standard solutions with tip p^k_{stan} and $x_k \in M^k_{\text{stan}}$ such that $\text{dist}_0(p^k_{\text{stan}}, x_k) \to \infty$*. Then a subsequence of the sequence of pointed Ricci flows $(M^k_{\text{stan}} \times I_0, (x_k, 0))$ smoothly converges to the standard round shrinking cylinder $(S^2 \times \mathbb{R}) \times I_0$.*
In the first case set $M^k_{\text{stan}} \times I^k := M_{\text{stan}} \times I$. Obviously we have smooth Gromov-Hausdorff convergence of the $(M^k_{\text{stan}}(0), x_k)$ to the standard round cylinder $S^2 \times \mathbb{R}$. By step 3 this convergence extends to the convergence of Ricci flows with limit $M_\infty \times I$ resp. $M_\infty \times I_0$. Since the limit is diffeomorphic to $S^2 \times \mathbb{R}$ and has nonnegative sectional curvature, we get by the Ricci splitting Theorem 1.4.3 and Corollary 2.5.7 that $M_\infty \times I = (N \times \mathbb{R}) \times I$ resp. $M_\infty \times I_0 = (N \times \mathbb{R}) \times I_0$ where $N \times I$ resp. $N \times I_0$ is a Ricci flow on the 2-sphere S^2 with the round starting metric hence N is round for all times I resp. I_0.

step 5: If $T_0 < 1$, then $I_0 = [0, T_0]$.
Assume that $I_0 = [0, T_0)$ and particularly $T_0 > 0$. Observe that by step 1 there is a universal $\kappa > 0$ such that all standard solutions are κ-noncollapsed. Using the canonical neighborhood assumptions and Lemma 6.2.2, it is easy to conclude that there is a $\kappa' > 0$ such that *the time slices* of all standard solutions are κ'-noncollapsed on scales < 1. We will first show that there is a radius d such that we can uniformly bound the curvature on balls $(M_{\text{stan}} \setminus B_d(p, 0)) \times I_0$ for all standard solutions $M_{\text{stan}} \times I$. If this were not the case, we could find a sequence of standard solutions $M^k_{\text{stan}} \times I^k$ with tips p^k_{stan} and points $(x_k, t_k) \in M^k_{\text{stan}} \times I^k$ such that $\text{dist}_0(p^k_{\text{stan}}, x_k) \to \infty$ and $S(x_k, t_k) \to \infty$ contradicting step 4 and Lemma 6.2.2. Now by the bounded curvature at bounded distances result, Proposition 6.2.4, and the fact that the metric shrinks we get global bounds on S. Thus we can extend all standard solutions until time T_0 and bound the curvature for all times $[0, T_0]$.

step 6: $I_0 = [0, 1)$.
Assume not. By step 5 we must have $I_0 = [0, T_0]$.

We first show that there is a $\tau > 0$ such that we can uniformly control the curvature at times $[0, T_0 + \tau] \cap I$ for all all standard solutions $M_{\text{stan}} \times I$. In the case $T_0 > 0$ this easily follows from step 2 and Lemma 6.2.2. If $T_0 = 0$, we have to make some more effort: Let $M_{\text{stan}} \times I$ be a standard solution. Using the first assertion of step 4, we conclude that for some large d we have $S(\cdot, t) < 2$ on $(M_{\text{stan}} \setminus B_d(p_{\text{stan}}, 0)) \times [0, \frac{1}{8}]$ (otherwise we could find a sequence of points $x_k \in M_{\text{stan}}$ and times $t_k \in [0, \frac{1}{8}]$ with $\text{dist}(p_{\text{stan}}, x_k) \to \infty$ and

$S(x_k, t_k) \geq 2$). Now we can apply the weak maximum principle on $\overline{B}_d(p_{\text{stan}}, 0)$ for times $[0, \frac{1}{4}]$ and conclude analogously to Corollary 2.5.5 that $S < 4$ on $B_d(p_{\text{stan}}, 0) \times [0, \frac{1}{8}]$.

Let C be the bound for the scalar curvature corresponding to the time interval $[0, T_0 + \tau]$. Set $\tau' := \min\{\tau, \frac{1}{4}C^{-1}\}$. We now show that any standard solution is defined for at least the times $[0, T_0 + \tau']$. Let $M_{\text{stan}} \times I$ be a standard solution with tip p_{stan} and assume $I \subsetneqq [0, T_0 + \tau']$. Then $I = [0, T]$ for some $T_0 \leq T \leq 1$. We want to extend the Ricci flow past T. Take a divergent sequence $x_k \in M^k_{\text{stan}}$. The sequence of pointed Ricci flows $(M_{\text{stan}} \times [0, T], (x_k, 0))$ subconverges by the results of step 4 to the standard round shrinking cylinder solution $(S^2 \times \mathbb{R}) \times [0, T]$. Thus $(M_{\text{stan}}(T), x_k)$ subconverges to $((S^2 \times \mathbb{R})(T), x_\infty)$ and we find regions in $M_{\text{stan}}(T)$ that are arbitrarily close to arbitrarily long cylinders of scalar curvature $\frac{1}{1-T}$. By cutting along central 2-spheres of these cylinders and gluing in a ball with standardized geometry, we can present $(M_{\text{stan}}(T), p_{\text{stan}})$ as a limit of compact, pointed 3-manifolds (M'_k, p'_k) such that the curvature derivatives $\nabla^l R$ are uniformly bounded and the manifolds satisfy the φ_k-positive curvature condition at time T with $\varphi_k \to 0$. By Theorems 2.7.1, 2.7.2 and Corollary 2.5.5 we can evolve the metric on the manifolds M'_k to obtain Ricci flows $M'_k \times [T, T + \tau']$ such that the scalar curvature is uniformly bounded by $2C$. Moreover by Theorem 2.9.1 the curvature is φ_k-positive and thus the Riemannian curvature is uniformly bounded on the $M'_k \times [T, T + \tau']$. Applying the weak maximum principle to the evolution equations of the $\|\nabla^l R\|^2$ (see (2.3)), we can uniformly bound the curvature derivatives on the $M'_k \times [T, T + \tau']$. Thus the sequence $(M'_k \times [T, T + \tau'], (p'_k, T))$ subconverges to a Ricci flow $(M_\infty \times [T, T + \tau'], (p'_\infty, T))$ of bounded nonnegative sectional curvature with starting metric $M_\infty(T) = M_{\text{stan}}(T)$. But this contradicts the fact that we cannot extend the Ricci flow $M_{\text{stan}} \times I$.

step 7: On any standard solution $M_{\text{stan}} \times I$ the sectional curvature is pointwise unbounded for $t \nearrow 1$ and thus $I = [0, 1)$.
Assume that for some $x \in M_{\text{stan}}$ the curvature $S(x, \cdot)$ is bounded on I. By the bounded curvature at bounded distances result, Proposition 6.2.4, we have pointwise bounds on S at times I everywhere (recall from step 5 that the time slices of $M_{\text{stan}} \times I$ are κ'-noncollapsed for some $\kappa' > 0$). So we can extend $M_{\text{stan}} \times I$ to time 1. We will now analyze the time 1 metric on M_{stan}. The result of step 4 gives us that for any $\delta > 0$ we find a sequence of times t_k and a divergent sequence points $x_k \in M_{\text{stan}}$ such the points (x_k, t_k) are centers of δ-necks whose widths converge to 0. Since the metric on $M_{\text{stan}} \times [0, 1]$ shrinks, this gives us subsets $U_k \subset M_{\text{stan}}$ whose diameter at time 1 goes to 0 and which separate M_{stan} into two pieces such that the part of $M_{\text{stan}} \setminus U_k$ that does not contain p_{stan} is noncompact. Now we can deduce a contradiction as in the proof of Corollary 5.1.9.

Conclusion
Assertions (a) and (d) are now clear and (e) follows from step 4. In order to prove (c) we observe first that for any $\delta, \theta > 0$ we must have $S \geq 1 - \delta$ outside some compact set $K \subset M_{\text{stan}}$ at times $[0, 1 - \theta)$. Otherwise we could find a divergent sequence of points $x_k \in M_{\text{stan}}$ with $S(x_k, t_k) < 1 - \delta$ for some t_k contradicting (e). Applying the weak maximum principle (see Theorem 2.5.1) to the evolution equation of S on K (see (2.2)), we conclude that the lower bound $1 - \delta$ holds everywhere on M_{stan} at times $[0, 1 - \theta]$. Now let $\delta, \theta \to 0$.

Finally, assertion (b) can be proved with the result of step 7, step 2 and Lemma 6.2.2 for times $t \in [1 - \frac{r^2}{\eta^2}, 1)$ and (c) for times $[0, 1 - \frac{r^2}{\eta^2}]$. □

We will now classify the approximate geometry of neighborhoods around points in standard solutions on a local scale. As in Theorem 5.4.11 we will be able to show that neighborhoods essentially look like necks or caps. However, we cannot expect that the necks are strong in the sense of Definition 5.4.4 since standard solutions are not ancient. For this we define:

Definition 7.3.3 (strong ε-neck until time t_1). *Let $\varepsilon > 0$, $M \times I$ be a Ricci flow, $U \subset M$ an open subset and $J = [t_1, t_2] \subset I$ a closed subinterval. We say that $U \times J$ is a* strong ε-neck until time t_1 *if there is a scaling factor $\lambda > 0$ such that after parabolically rescaling the flow on $U \times J$ by the factor λ^{-1}, there is a (bijective) diffeomorphism $\Phi : S^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \to U$ that is an ε-isometry between the time t metric of the standard round shrinking cylinder on $S^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$ and the time $t + \lambda^{-2} t_2$ metric on $\lambda^{-1}(M \times J)$ for all $t \in [-\lambda^{-2}(t_2 - t_1), 0]$.*
Moreover, we call $(x, t_2) \in U \times \{t_2\}$ a center of $U \times J$ *if $x \in \Phi(S^2 \times \{0\})$ for such a Φ.*

Now we can formulate the result.

Theorem 7.3.4. *There are $\eta' < \infty$ and $\kappa_0' > 0$ and for every $\varepsilon > 0$ there is an $E''(\varepsilon) < \infty$ such that we have the following classification:*
If (x, t) is a point in a standard solution $M_{\text{stan}} \times [0, 1)$, then
(a) (x, t) is the center of a strong ε-neck or
(b) (x, t) is the center of an (ε, E'')-cap or
(c) (x, t) is the center of a strong ε-neck until time 0.
Moreover, $M_{\text{stan}}(t)$ is κ_0'-noncollapsed and

$$\|\nabla S^{-1/2}(x, t)\| < \tfrac{1}{2}\eta' \qquad \text{and} \qquad |\partial_t S^{-1}(x, t)| < \left(\tfrac{1}{2}\eta'\right)^2 \tag{7.3}$$

Proof. By the Canonical Neighborhood Theorem 6.3.2 and Theorem 7.3.2 (b) we know that there is some $\theta(\varepsilon) > 0$ such that for $t > 1 - \theta$ case (a) or (b) applies. For times $t \leq 1 - \theta$ we can use Lemma 7.3.2 (e) to conclude that cases (a) or (c) apply if $\text{dist}_0(p_{\text{stan}}, x)$ is larger than some constant depending only on ε. Since we have universal bounds on the geometry $K \times [0, 1 - \theta]$ for any compact set $K \subset M_{\text{stan}}$ (this includes control over the curvature derivatives, see step 3 in the proof of Lemma 7.3.2), we can choose E'' so large that the remaining points satisfy (b). The last two assertions follow immediately. $\square$

Observe that if we increase the constants η or η' from Corollary 5.2.9 resp. Theorem 7.3.4, all preceding assertions involving one of these two constants still hold true. So at this point, we may assume that $\eta = \eta'$. Analogously, we may assume that $\kappa_0 = \kappa_0'$ and that the theorems and lemmas involving these constants stay true.

In order to extend strong necks over a singularity time, we prove the following

Lemma 7.3.5. *For any $\varepsilon > 0$ there is a $0 < \overline{\varepsilon}(\varepsilon) < \varepsilon$ such that the following statement holds:*
Consider a Ricci flow with surgery $\mathcal{M}$. Suppose that $(x, t_2) \in \mathcal{M}$ is the center of a strong $\overline{\varepsilon}$-neck $U_2 \times [t_1, t_2]$ until time t_1 and (x, t_1) is the center of a strong $\overline{\varepsilon}$-neck $U_1 \times [t_0, t_1]$.
Then $(x, 0)$ is the center of a strong ε-neck.

Proof. Assume that this was wrong. Choose a sequence $\overline{\varepsilon}^k \to 0$ and find counterexamples $U_1^k \times [t_0^k, t_1^k], U_2^k \times [t_1^k, t_2^k] \subset \mathcal{M}^k$ and $(x^k, t_2^k) \in \mathcal{M}^k$. By shifting in time and parabolic rescaling we may assume that $t_2^k = 0$ and $S(x^k, 0) = 1$. Since $(x^k, 0)$ is not the center of a strong ε-neck, the t_1^k are bounded from below. After choosing a subsequence, we may assume that $t_1^k \to t_1^\infty \leq 0$.

Since $\overline{\varepsilon}_k \to 0$, we have uniform bounds for the curvature and its derivatives on $U_2^k \times [t_1^k, 0]$. So the scaling factor λ^k of the $\overline{\varepsilon}^k$-neck $U_1^k \times [t_0^k, t_1^k]$ is bounded from below. This implies that $B_{d^k}(x^k, 0) \subset U_1^k, U_2^k$ for a certain sequence $d^k \to \infty$ and that $t_0^k \to -\infty$. Moreover, we have uniform time dependent bounds for the curvature and its derivatives on $P(x^k, 0, d^k, t_0^k)$.

So we have subconvergence of the $(P(x^k, 0, d^k, t_0^k), (x^k, 0))$ to a Ricci flow $(M^\infty \times (-\infty, 0], (x^\infty, 0))$ that is isometric to the standard round shrinking cylinder at times $(-\infty, t_1^\infty]$ and $[t_1^\infty, 0]$, hence it is isometric to the standard round shrinking cylinder $(S^2 \times \mathbb{R}) \times (-\infty, 0]$. A contradiction. $\square$

7.4 Ricci flows with surgery under canonical neighborhood assumptions

For the rest of this exposition we fix some constant $\varepsilon < \varepsilon_0$ and set $\overline{\varepsilon} := \overline{\varepsilon}(\frac{\varepsilon}{2}) < \varepsilon$ as in Lemma 7.3.5. Furthermore set

$$E_0 := \max\left\{4E_{\kappa_0}\left(\frac{\varepsilon}{4}\right), 4E''\left(\frac{\overline{\varepsilon}}{4}\right)\right\}$$

where E denotes the constant from Theorem 5.4.11, κ_0 the constant from Theorem 5.4.13 and E'' the constant from Theorem 7.3.4.

Now let $0 < r < 1$ and $\delta > 0$ be arbitrary constants. In this section we study Ricci flows with surgery $\mathcal{M}$ that satisfy the following properties:

(i) $\mathcal{M}$ is defined on some time interval $I = [0, T)$ or $[0, T]$ and has normalized initial conditions.

(ii) At every surgery time the surgeries are performed by $(r'(t), \delta'(t))$-cutoff, where $r'(t)$ and $\delta'(t)$ are constants depending on time such that $0 < r' < 1$ and $0 < \delta' < \delta$. We assume furthermore, that for every surgery time t the canoncial neighborhood assumptions $\mathrm{CNA}(r'(t), \varepsilon, E_0)$ hold at all non-presurgery points on $[0, t)$ to make the surgery step possible.

(iii) The canonical neighborhood assumptions $\mathrm{CNA}(\frac{1}{2}r, 2\varepsilon, 2E_0)$ hold at all non-presurgery points.

The first Lemma gives a description of how neighborhoods near surgeries look like.

Lemma 7.4.1. *Let $(p, t) \in \mathcal{M}$ be the tip of a surgery and $A < \infty$, $\theta, \varphi > 0$. If $\delta < \delta_1(A, \theta, \varphi)$ then there is some $0 \le \sigma \le 1 - \theta$ and a standard solution $M_{\mathrm{stan}} \times [0, 1)$ (with tip p_{stan}) such that there is a φ-homothety*

$$M_{\mathrm{stan}} \times [0,1) \supset B(p, 0, A) \times [0, \sigma) \xrightarrow{\Phi} \mathcal{M} \qquad \text{with} \qquad (p_{\mathrm{stan}}, 0) \mapsto (p, t)$$

whose image only meets surgery points at its first time slice.

Furthermore, if $\sigma < 1 - \theta$, then no point of $\Phi(B_A(p, 0) \times \{0\})$ survives past the final time of $\overline{\operatorname{Im}\Phi}$ (this may be due to surgeries or to the fact that the final time of $\mathcal{M}$ is reached).

Observe that δ_1 does not depend on r.

Proof. Fix A, θ, φ. Assume that the assertion was wrong. Choose a sequence $\delta_k \to 0$ and consider counterexamples $\mathcal{M}_k$ for each δ_k together with constants $0 < r_k < 1$ and tips of surgeries $(p_k, t_k) \in \mathcal{M}_k$. Observe that $S(p_k, t_k) \ge \frac{1}{\delta_k^2 r_k^2} \to \infty$. Shift the $\mathcal{M}_k$ in time and rescale to get $\mathcal{M}'_k$ such that for the corresponding tips of surgeries $(p'_k, 0) \in \mathcal{M}'_k$ we have $S(p'_k, 0) = S_0$. Then the $\mathcal{M}'_k$ have $S_0 S^{-1}(p_k, t_k)$-positive curvature.

Observe that the time 0^- slices $\mathcal{M}'_k(0^-)$ contain the final time slices V'_k of strong δ_k-necks whose metric will be changed on one half during the cutoff process and remain the same on the other. Let $V_k \subset \mathcal{M}'(0^+)$ be the open sets that come out of the V'_k after the cutoff (i.e. the union of the half that has not been changed and the part that carries the new metric). For large k we can assume $B_{\frac{1}{2}\delta_k^{-1}}(p'_k, 0) \subset V_k$. Together with the remarks in section 7.2, we find that the pointed manifolds (V_k, p'_k) smoothly converge to the standard cap $(M_{\mathrm{stan}}, p_{\mathrm{stan}})$.

Let $\sigma_0 \in [0,1]$ be maximal with the property that after passing to a subsequence there are sequences $d_k \to \infty$ and $\tau_k \to 0$ such that the parabolic neighborhoods $P(p'_k, 0, d_k, \sigma_0 - \tau_k)$ are non-singular and we have convergence

$$(P(p'_k, 0, d_k, \sigma_0 - \tau_k), (p'_k, 0)) \xrightarrow[k\to\infty]{} (M_\infty \times [0, \sigma_0), (p_\infty, 0)) . \tag{7.4}$$

Obviously, the sectional curvature on $M_\infty \times [0, \sigma_0)$ is nonnegative and $M_\infty(0) = M_{\text{stan}}$. So $M_\infty \times [0, \sigma_0)$ is the restriction of a standard solution $M_{\text{stan}} \times [0, 1)$ to the time interval $[0, \sigma_0)$. If $\sigma_0 > 1 - \theta$, then the Lemma follows immediately.

Suppose now that $\sigma_0 \le 1 - \theta$. By Lemma 7.3.2 (d) we have a uniform bound C on the scalar curvature of $M_\infty \times [0, \sigma_0)$. Let $\tau > 0$ be a small constant that we will specify later. Assume that for a subsequence there is a sequence $d'_k \to \infty$ such that $P(p'_k, 0, d'_k, \sigma_0 + \tau)$ is non-singular. We may assume $d'_k < \frac{1}{2}\delta_k^{-1}$. Then for sufficiently small τ we can use Lemma 6.2.2 to bound the curvature uniformly on these parabolic neighborhoods. Analogously to the proof of step 3 in Theorem 7.3.2 this bound implies bounds for the curvature derivatives $\nabla^l R$: The local Shi estimates give bounds for the $\nabla^l R$ on $V_k \setminus B_{\frac{1}{2}\delta_k^{-1}}(p'_k, 0)$ since the metric remained unchanged on this set during the surgery process. On $\overline{B}_{\frac{1}{2}\delta_k^{-1}}(p'_k, 0)$ we can apply the weak maximum principle to the evolution equations of the $\|\nabla^l R\|^2$.

So we can find a subsequence such that the convergence (7.4) applies for σ_0 replaced by $\sigma_0 + \tau$, d_k replaced by d'_k and $\tau_k = 0$. A contradiction to the maximality of σ_0. Thus we conclude that there is a $d < \infty$ such that $P(p'_k, 0, d, \sigma_0 + \tau)$ is singular for large k. Set $A' := \max\{A, d\}$.

For any k let σ_k be supremal with the property that $P(p'_k, 0, A', \sigma_k)$ is non-singular. Obviously, $\sigma_k \le \sigma_0 + \tau$ for large k and $\liminf_{k\to\infty} \sigma_k \ge \sigma_0$. So there are points $x_k \in B_{A'}(p'_k, 0)$ that do not survive past time σ_k. Assume that for some k there is a point $y_k \in B_{A'}(p'_k, 0)$ that does. Then there is a point $z_k \in B_{A'}(p'_k, 0)$ that is the center of a strong δ_k-neck at time σ_k at which a surgery is performed. If τ is small enough, we have a uniform upper bound on the curvature in (z_k, σ_k). So we can bound the length and the living time of this strong δ_k-neck from below in terms of δ_k. But this implies that the point $(p'_k, 0)$ must lie in the interior of this strong neck for large k. A contradiction. So for large k no point in $B_{A'}(p'_k, 0)$ survives past time σ_k.

Finally, we have to show that the solutions on $P(p'_k, 0, A, \sigma_k) \backslash B_A(p'_k, \sigma_k)$ get arbitrarily close to the solution on $B_A(p_{\text{stan}}, 0) \times [0, \sigma_k) \subset M_{\text{stan}} \times [0, \sigma_k)$ if k is sufficiently large and τ sufficiently small. This is clear, if $\sigma_k \le \sigma_0$. If $\sigma_k > \sigma_0$, observe that $\sigma_k \le \sigma_0 + \tau$ for large k and thus for τ sufficiently small, the metrics on $P(p'_k, 0, A, \sigma_k) \setminus P(p'_k, 0, A, \sigma_0)$ and $B_A(p_{\text{stan}}, 0) \times [\sigma_0, \sigma_k)$ are sufficiently close to the metric on their first time slice (in the smooth sense) to conclude the desired result. Recall hereby that we have uniform bounds on the curvature derivatives $\nabla^l R$. $\square$

Let $(x_0, t_0) \in \mathcal{M}$ be a point such that t_0 is larger than or equal to the last surgery time. If t_0 is equal to the last surgery time, we consider t_0 as a postsurgery time. The next Lemma asserts that if (x_0, t_0) has nearby surgeries, then it already has a canonical neighborhood.

Lemma 7.4.2. *For every $\theta > 0$ and $A < \infty$ and if $\delta < \delta_2(A, \theta)$, we have the following: Assume that*

(i) the point $(x, t) \in \mathcal{M}$ is a surgery point and let $(p, t) \in \mathcal{M}$ be the tip of the corresponding surgery.
(ii) $t_0 - t \le (1-\theta) S_0 S^{-1}(p, t)$.
(iii) the point (x, t) survives (at least) until time t_0.
(iv) $\operatorname{dist}_{t_0}(x, x_0) < A S_0^{1/2} S^{-1/2}(p, t)$.

Then (x_0, t_0) satisfies the canonical neighborhood assumptions $\mathrm{CNA}(2r, \frac{\varepsilon}{2}, \frac{E_0}{2})$.

Proof. Observe first that by the way the surgeries are performed there is a universal constant d_0 such that $\mathrm{dist}_t(p,x) < d_0 S_0^{1/2} S^{-1/2}(p,t)$. Let C be a universal bound for the distance distortion on $M_{\mathrm{stan}} \times [0, 1-\frac{\theta}{2}]$ on any standard solution $M_{\mathrm{stan}} \times [0,1)$ (see Theorem 7.3.2 (d) and 2.3.1). That is to say $\frac{1}{C}\,\mathrm{dist}_{t_1} \le \mathrm{dist}_{t_2} \le C\,\mathrm{dist}_{t_1}$ for any $t_1, t_2 \in [0, 1-\frac{\theta}{2}]$. Set $A' := 2d_0 + 2AC + \max\{E_0^2, 2\varepsilon^{-1}\}C$.

In the course of the proof we will choose δ_2 smaller and smaller. It will be easy to check that δ_2 only depends on A and θ. Assume first that $\delta_2 < \delta_1(A', \frac{1}{2}\theta, \varphi)$ where δ_1 denotes the constant from Lemma 7.4.1 and $\varphi > 0$ a constant that we will specify later. By Lemma 7.4.1 there is a standard solution $M_{\mathrm{stan}} \times [0,1)$ with tip p_{stan} and a φ-homothety with scaling factor λ

$$M_{\mathrm{stan}} \times [0,1) \supset B_{A'}(p_{\mathrm{stan}}, 0) \times [0,\sigma) \xrightarrow{\Phi} \mathcal{M} \qquad \text{with} \qquad (p_{\mathrm{stan}}, 0) \mapsto (p,t)$$

for some $0 \le \sigma \le 1 - \frac{1}{2}\theta$. For sufficiently small φ we can assume that $\lambda S_0^{-1/2} S^{1/2}(p,t)$ is arbitrarily close to 1. Since x survives until time t_0, we conclude from (ii) that if φ is sufficiently small, t_0 is not larger than the final time of $\overline{\mathrm{Im}\,\Phi}$, i.e. $t_0 \le t + \lambda^2\sigma$. Let $(x', t') \in M_{\mathrm{stan}} \times [0,1)$ be the point corresponding to (x_0, t_0) under Φ (if $t' = \sigma$ this means $\lim_{s \to t'} \Phi(x', s) = (x_0, t_0)$). Choosing φ sufficiently small, we may assume that $(x', t') \in P(p_{\mathrm{stan}}, 0, 2d_0 + 2AC, 1 - \frac{\theta}{2})$.

Assume first that the balls $B_{E_0^2 S^{-1/2}(x_0,t_0)}(x_0, t_0)$ and $B_{4\varepsilon^{-1} S^{-1/2}(x_0,t_0)}(x_0,t_0)$ do not contain any postsurgery points. By Theorem 7.3.4 the point (x', t') is the center of a strong $\frac{\varepsilon}{4}$-neck, an $(\frac{\varepsilon}{4}, \frac{E_0}{4})$-cap or a strong $\frac{\overline{\varepsilon}}{4}$-neck until time 0. Since the scalar curvature on $M_{\mathrm{stan}} \times [0,1)$ is uniformly bounded from below by 1, the scaling factor of these necks can be assumed to be less than 2. Now we may conclude that if φ is smaller than some universal constant, the point (x_0, t_0) must be the center of a strong $\frac{\varepsilon}{2}$-neck, an $(\frac{\varepsilon}{2}, \frac{E_0}{2})$-cap or a strong $\frac{\overline{\varepsilon}}{2}$-neck until time t. Observe that here we have used the following fact: For every φ_2 and $\lambda > 0$ there is a $\varphi_1(\varphi_2, \lambda) > 0$ that is nonincreasing in λ such that if $\Psi : N_1 \to N_2$ is a φ_1-isometry between two Riemannian manifolds (N_1, g_1) and (N_2, g_2) then Ψ is a φ_2-isometry between the rescaled manifolds $\lambda^{-1} N_1$ and $\lambda^{-1} N_2$.

Now suppose that $B_{E_0^2 S^{-1/2}(x_0,t_0)}(x_0,t_0)$ or $B_{4\varepsilon^{-1} S^{-1/2}(x_0,t_0)}(x_0,t_0)$ does contain a postsurgery point (x^*, t_0). Denote by (p^*, t_0) the tip of the corresponding surgery cap. Observe that by the fact that $(x_0, t_0) \in \overline{\mathrm{Im}\,\Phi}$ there is a (universal) C' such that $S(p^*, t_0) < C' S(x_0, t_0)$. Moreover, we can estimate $\mathrm{dist}_{t_0}(x^*, x_0) S_0^{-1/2} S^{1/2}(p^*, t_0)$ from above by a universal constant. Now assume δ_2 to be sufficiently small such that we can repeat the preceding arguments for (x^*, t_0) instead of (x,t) and (p^*, t_0) instead of (p,t). Thus we can still assume that in this case (x_0, t_0) is the center of an $(\frac{\varepsilon}{2}, \frac{E_0}{2})$-cap or a strong $\frac{\overline{\varepsilon}}{2}$-neck until time t_0 (in the latter case the strong neck until time t_0 is actually a neck).

In the case in which (x_0, t_0) is the center of a strong $\frac{\varepsilon}{2}$-neck or an $(\frac{\varepsilon}{2}, \frac{E_0}{2})$-cap, we are done. Assume now that (x_0, t_0) is the center of a strong $\overline{\varepsilon}$-neck until time t resp. t_0. Now consider the point $(x_0, t^-) \in \mathcal{M}(t^-)$ resp. $(x_0, t_0^-) \in \mathcal{M}(t_0^-)$. Since (x_0, t^+) lies close to (p, t^+) resp. (p^*, t_0^+), the point (x_0, t^-) resp. (x_0, t_0^-) has to lie close to the center of a strong δ-neck. In particular for sufficiently small δ we get that (x_0, t^-) resp. (x_0, t_0^-) is the center of a strong $\overline{\varepsilon}$-neck and by Lemma 7.3.5 we conclude that (x_0, t_0) is the center of a strong $\frac{\varepsilon}{2}$-neck.

It is now easy to show that (x_0, t_0) also satisfies properties (A) and (B) of the canonical neighborhood assumptions $\mathrm{CNA}(2r, \frac{\varepsilon}{2}, \frac{E_0}{2})$ (use the last two assertions of Theorem 7.3.4). $\square$

Assume now, that (x_0, t_0) does not satisfy $\mathrm{CNA}(2r, \frac{\varepsilon}{2}, \frac{E_0}{2})$. As a preparation for subsection 7.5 we discuss a tool that will help us to exclude surgery points in larger and larger parabolic neighborhoods around (x_0, t_0).

Lemma 7.4.3. *There are numbers $\alpha, \beta > 0$ such that under the above assumptions we have:*
Assume that (x_0, t_0) does not satisfy $\mathrm{CNA}(2r, \frac{\varepsilon}{2}, \frac{E_0}{2})$. Let $Q := S(x_0, t_0)$.

(i) *For any $a, K < \infty$ there is a $\delta_3(a, K)$ such that the following holds:*
If $\delta < \delta_3$ and $S \leq KQ$ on $B_{aQ^{-1/2}}(x_0, t_0)$ and if the ball doesn't contain surgery points, then $P(x_0, t_0, (a + \alpha K^{-1/2})Q^{-1/2}, -\beta K^{-1} Q^{-1})$ doesn't contain surgery points either.

(ii) *For any $a, b, K < \infty$ there is a $\delta_4(a, b, K)$ such that the following holds:*
If $\delta < \delta_4$ and $S \leq KQ$ on $P(x_0, t_0, aQ^{-1/2}, -bQ^{-1})$ and if this parabolic neighborhood doesn't contain surgery points, then $P(x_0, t_0, aQ^{-1/2}, -(b + \beta K^{-1})Q^{-1})$ doesn't contain any surgery points either.

Proof. Observe that $Q = S(x_0, t_0) > \frac{1}{4r^2}$. So the canonical neighborhood assumptions $\mathrm{CNA}(\frac{1}{4}Q^{-1/2}, 2\varepsilon, 2E_0)$ hold at all non-presurgery points. Without loss of generality we may assume that $K \geq 16$. Set

$$\alpha := \frac{1}{2\eta}, \qquad \beta := \min\left\{\frac{1}{8\eta^2}, \frac{1}{32}\right\}.$$

(i) For $A := 4(a\sqrt{K} + \alpha)$ assume that $\delta_4 < \delta_2(A, \frac{1}{2})$ where δ_2 denotes the constant obtained from Lemma 7.4.2. Moreover, assume δ_4 to be so small such that for any surgery point $(x, t^+) \in \mathcal{M}$ and tip (p, t) of the corresponding surgery cap we have $S(p, t) < 2S_0 S(x, t^+)$.

Assume that the hypothesis was wrong. Then there is a post-surgery point $(x, t^+) \in P(x_0, t_0, (a + \alpha K^{-1/2})Q^{-1/2}, -\beta K^{-1/2} Q^{-1/2})$ that survives until time t_0. Let (p, t) be the tip of the corresponding surgery cap. We will first estimate $S(p, t)$ from above. From Lemma 6.2.1 we get

$$S^{-1/2}(x, t_0) > (KQ)^{-1/2} - \eta\alpha (KQ)^{-1/2} = \frac{1}{2}(KQ)^{-1/2}$$

and Lemma 6.2.2 gives

$$S^{-1}(x, t^+) > S^{-1}(x, t_0) - \eta^2 \beta (KQ)^{-1} > \frac{1}{4}(KQ)^{-1} - \eta^2\beta(KQ)^{-1} \geq \frac{1}{8}(KQ)^{-1}.$$

So $S(p, t) < 16 S_0 KQ$.

Since $t_0 - t \leq \beta(KQ)^{-1} \leq \frac{1}{32}(KQ)^{-1} < \frac{1}{2} S_0 S^{-1}(p, t^+)$ and $\mathrm{dist}_{t_0}(x, x_0) < (a + \alpha K^{-1/2})Q^{-1/2} = A(16KQ)^{-1/2} < AS_0^{-1/2} S^{-1/2}(p, t)$ we may apply Lemma 7.4.2 to find that (x_0, t_0) already has a canonical neighborhood. A contradiction.

(ii) Choose d_0 so large that for any surgery point $(x, t) \in \mathcal{M}$ and tip of the corresponding surgery (p, t) we have $\mathrm{dist}_t(x, p) < d_0 S_0^{1/2} S^{-1/2}(p, t)$. Set

$$\theta_1 := \frac{c}{16(bK + \beta)}, \quad \theta_2 := \min\left\{\frac{\theta_1}{2}, \frac{1}{2}\right\}, \quad \mu := \min\left\{\frac{1 - \theta_2}{1 - \theta_1}, 2\right\} > 1,$$

$$A_1 := 2d_0, \quad A_2 := a\sqrt{2K}$$

and choose $\varphi_1 > 0$ so small such that any φ_1-isometry $\Phi : N \to N'$ is 2-Lipschitz and distorts the scalar curvature by a factor of at most μ, i.e. $\frac{1}{\mu}S < S \circ \Phi < \mu S$. Now set $\delta_4(a, b, K) := \min\{\delta_1(A_1, \theta_1, \varphi_1), \delta_2(A_2, \theta_2)\}$ where δ_1, δ_2 denote the constants obtained in Lemma 7.4.1 and 7.4.2.

Choose $0 \leq \tau \leq \beta(KQ)^{-1}$ maximal with the property that there are no surgeries in the parabolic neighborhood $P := P(x_0, t_0, aQ^{-1/2}, -bQ^{-1} - \tau)$ except maybe at its time $T - bQ^{-1} - \tau$ slice. By Lemma 6.2.2 we have on P

$$S^{-1} \geq (KQ)^{-1} - \beta(KQ)^{-1}\eta^2 > \frac{1}{2}(KQ)^{-1} \qquad \Longrightarrow \qquad S < 2KQ.$$

Now assume that there is a surgery point $(x, t) \in P$ with $t = T - bQ^{-1} - \tau$. Let (p, t) be the tip of the corresponding surgery cap. We have $S(p, t) \leq 2KQ$ and thus

$$\operatorname{dist}_{t_0}(x, x_0) < aQ^{-1/2} = A_2(2KQ)^{-1/2} < A_2 S_0^{-1/2} S^{-1/2}(p, t).$$

An application of Lemma 7.4.1 with parameters A_1, θ_1 and φ_1 to the point (p, t) shows us that there is $0 \leq \sigma \leq 1 - \theta_1$, a standard solution $M_{\mathrm{stan}} \times [0, 1)$ and a φ_1-homothety with scaling factor λ

$$M_{\mathrm{stan}} \times [0, 1) \supset B_A(p_{\mathrm{stan}}, 0) \times [0, \sigma) \xrightarrow{\Phi} \mathcal{M} \qquad \text{with} \qquad (p_{\mathrm{stan}}, 0) \to (p, t).$$

By the choice of φ_1 we have $\frac{S_0}{\lambda^2} > \frac{1}{\mu} S(p, t)$.

Observe that the point (x, t^+) survives until time t_0. So if $\sigma < 1 - \theta_1$, the final time of $\operatorname{Im}\Phi$ must be not less than t_0 and

$$t_0 - t \leq \sigma\lambda^2 < (1 - \theta_1)\, \mu S_0 S^{-1}(p, t) \leq (1 - \theta_2)\, S_0 S^{-1}(p, t).$$

Now Lemma 7.4.2 gives a contradiction.

On the other hand suppose that $\sigma = 1 - \theta_1$. By Theorem 7.3.2 (b) the scalar curvature on $M_{\mathrm{stan}} \times [0, 1)$ at the time $1 - \theta_1$ slice is larger than $\frac{c}{\theta_1} = 16(bK + \beta)$. So for $t' := t + \lambda^2\theta_1 \in [t, t_0]$ we get

$$\frac{1}{\mu} \cdot 16(bK + \beta) \cdot \frac{1}{\lambda^2} < S(x, t') < 2KQ \qquad \Longrightarrow \qquad (b + \beta K^{-1})Q^{-1} < \frac{1}{4}\lambda^2.$$

Hence

$$\begin{aligned} t_0 - t \leq (b + \beta K^{-1})Q^{-1} < \frac{1}{4}\lambda^2 < \frac{\mu}{4} S_0 S^{-1}(p, t) \\ < \frac{1}{2} S_0 S^{-1}(p, t) \leq (1 - \theta_2)\, S_0 S^{-1}(p, t). \end{aligned}$$

Applying Lemma 7.4.2 yields the desired contradiction. □

7.5 Justification of the canonical neighborhood assumptions

Consider again Ricci flows with surgery $\mathcal{M}$ defined on $[0, T)$ that have normalized initial conditions. Let $I \subset [0, \infty)$ be an interval. $\mathcal{M}$ is said to satisfy some canonical neighborhood assumptions on I if these assumptions hold at all points at times $I \cap [0, T)$. Analogously we define what it means to say that $\mathcal{M}$ is κ-noncollapsed on I. Let $\delta, r : [0, T) \to \mathbb{R}^+$ be functions. We say that $\mathcal{M}$ is a Ricci flow with $(\delta(t), r(t))$-*cutoff* if at any surgery time t the surgeries are performed by $(\delta(t), r(t))$-cutoff.

We will now show that if we perform the $(\delta(t), r(t))$-cutoff appropriately (this will mean that $r(t)$ and $\delta(t)$ is smaller than some time dependent constants), every non-presurgery point satisfies some canonical neighborhood assumptions with constants that only depend on time and neither on the initial metric nor the number of surgeries performed so far. With this result it is clear that the construction of $\mathcal{M}$ as described in section 7.2 works and that we can control the parameters $r(t)$ and $\delta(t)$ sufficiently well to ensure that the surgery times cannot accumulate.

Theorem 7.5.1. *There are sequences of positive numbers $r_1 > r_2 > \ldots$, $\kappa_1 > \kappa_2 > \ldots$ and $\delta_1 > \delta_2 > \ldots$ such that the following holds:*
Let $0 < \delta(t) \leq \delta_j$ and $0 < r(t) \leq r_j$ on $[2^{j-1}, 2^j]$ (resp. $[0,2]$ for $j = 1$). Assume that $\mathcal{M}$ is a 3 dimensional Ricci flow with surgery defined on a time interval $[0,T)$ with the following properties:

(i) $\mathcal{M}$ has normalized initial conditions,
(ii) the surgeries are performed by $(\delta(t), r(t))$-cutoff and
(iii) for every surgery time $t \in [0,T)$ the canonical neighborhood assumptions $\mathrm{CNA}(r(t), \varepsilon, E_0)$ hold at all non-presurgery points on $[0,t)$.

Then for all $j \geq 1$

(a) $\mathcal{M}$ is κ_j-noncollapsed on $[0, 2^j]$ and
(b) the canonical neighborhood assumptions $\mathrm{CNA}(r_j, \varepsilon, E_0)$ hold at all non-presurgery points on $[0, 2^j]$.

Lemma 7.5.2. *Let $i \geq 0$ and $\kappa_1 > \ldots > \kappa_i > 0$, $r_1 > \ldots > r_{i+1} > 0$ and $\delta_1 > \ldots > \delta_{i-1} > 0$. Then we can find some constants $\kappa_{i+1}, \delta > 0$ such that:*
Let $\delta_i, \delta_{i+1} \leq \delta$ (resp. $\delta_{i+1} \leq \delta$ if $i = 0$), $0 < \delta(t) \leq \delta_j$ and $0 < r(t) \leq r_j$ on $[2^{j-1}, 2^j]$ (resp. $[0,2]$ for $j = 1$). Assume that $\mathcal{M}$ is a 3 dimensional Ricci flow surgery defined for times $[0,T)$ with $T \leq 2^{i+1}$ that satisfies (i)-(iii) (in Theorem 7.5.1) and

(iv) $\mathcal{M}$ satisfies (a) and (b) for $j = 1, \ldots, i$ and
(v) the canonical neighborhood assumptions $\mathrm{CNA}(\frac{1}{2} r_{i+1}, 2\varepsilon, 2E_0)$ hold at all non-presurgery points on $[0, 2^{i+1}]$.

Then $\mathcal{M}$ satisfies (a) for $j = i+1$.

We will prove this Lemma later.

Proof of the theorem. By induction we may assume that we already have sequences $r_1 \geq \ldots \geq r_i$, $\kappa_1 \geq \ldots \geq \kappa_i$ and $\delta_1 \geq \ldots \geq \delta_i$ for $i \geq 0$ such that (a) and (b) apply for all $j = 1, \ldots, i$.

Choose first $r_{i+1} < r_i$ arbitrarily and determine κ_{i+1} and δ for $r_1, \ldots, r_i, r_{i+1}, \kappa_1, \ldots, \kappa_i$ and $\delta_1, \ldots, \delta_{i-1}$ from Lemma 7.5.2. Set $\delta_{i+1} := \delta$ replace δ_i by $\min\{\delta_i, \delta\}$ (if $i > 0$). Let $\delta(t), r(t)$ be functions with $0 < \delta(t) \leq \delta_j$ and $0 < r(t) \leq r_j$ on $[2^{j-1}, 2^j]$ for all $j > 1$ (resp. $[0,2]$ for $j = 1$). Consider a 3 dimensional Ricci flow with surgery $\mathcal{M}$ defined on some time interval $[0,T) \subset [0, 2^{i+1}]$ that satisfies (i)-(iii). Recall that this implies that (a) and (b) are satisfied for all $j = 1, \ldots, i$. Let $t_0 \in [0, 2^{i+1}]$ be maximal with the property that the canonical neighborhood assumptions $\mathrm{CNA}(r_{i+1}, \varepsilon, E_0)$ hold at all non-presurgery points on $[0, t_0)$. Then $t_0 \geq 2^i$ for $i > 0$ and in the case $i = 0$ we have $t_0 \geq \frac{1}{4}$ if $r_1 < \frac{1}{\sqrt{2}}$ since by Corollary 2.5.5 the scalar curvature on $\mathcal{M}$ is bounded from above by 2 at times $[0, \frac{1}{4}]$.

We will first show that all points on the time t_0 slice (resp. t_0^+ slice if t_0 is a surgery time) satisfy the more general assumptions $\mathrm{CNA}(\frac{1}{2} r_{i+1}, 2\varepsilon, 2E_0)$ if $\delta_{i+1} < \delta'$ (where $\delta' > 0$ denotes some universal constant). If t_0 is not a surgery time, this is clear. So assume that t_0 is a surgery time. Let $x \in \mathcal{M}(t_0^+)$ with $S(x, t_0^+) > \frac{4}{r_{i+1}^2}$. If x is a surgery point, we get from Lemma 7.4.2 that (x, t_0^+) even satisfies the canonical neighborhood assumptions $\mathrm{CNA}(2r_{i+1}, \frac{\varepsilon}{2}, \frac{E_0}{2})$ (observe that we have a universal bound for the diameter of the glued-in caps on the local scale). If x is not a surgery point, choose $t' < t_0$ sufficiently close to t_0 such that we can apply the following reasoning: We have $S(x, t') > \frac{1}{r_{i+1}^2}$, so (x, t') is the center of a strong ε-neck or an (ε, E_0)-cap. If this neck or cap hits surgery points at time t_0^- we can again use Lemma 7.4.2 to conclude that (x, t_0^+) already has a canonical neighborhood for δ' sufficiently small (we have control over the curvature on the neck or cap and its diameter on a local scale). If not, (x, t_0^+) is the center of a strong 2ε-neck or a $(2\varepsilon, 2E_0)$-neck. Property (A) of the canonical neighborhood assumptions follows immediately For property (B) observe that in order to prove the required noncollapsedness we just have to consider the volumes of the balls that lie inside the time t_0^+ neck or cap (the curvature can be controlled from below, so the radii can be controlled from above).

Observe that by the choice of δ, Lemma 7.5.2 asserts that $\mathcal{M}$ satisfies (a) for $j = i+1$ on $[0, t_0]$.

Now assume that (b) does not hold for $j = i+1$. By the choice of t_0 this implies that there is a non-presurgery point $(x, t_0) \in \mathcal{M}$ (resp. $(x, t_0^+) \in \mathcal{M}$) that does not satisfy $\mathrm{CNA}(2r_{i+1}, \frac{1}{2}\varepsilon, \frac{1}{2}E_0)$ (otherwise we could conclude that $\mathrm{CNA}(r_{i+1}, \varepsilon, E_0)$ holds at all non-presurgery points on $[0, t_0 + \theta)$ for some $\theta > 0$ contradicting the maximality of t_0). We will show that this leads to a contradiction for r_{i+1} small enough.

Assume that we cannot find an appropriate r_{i+1}. Choose a sequence $r_{i+1}^k \to 0$. For every k we find δ^k as above and we may assume that $\delta^k \to 0$, particularly $\delta^k < \min\{\delta', \delta_i\}$ (otherwise replace the δ^k by even smaller numbers). Set $\delta_1^k := \delta_1, \ldots, \delta_{i-1}^k := \delta_{i-1}$ and $\delta_i^k = \delta_{i+1}^k := \delta^k$ (if $i = 0$ we just set $\delta_1^k := \delta^k$). We find a sequence of functions $\delta_k(t), r_k(t)$ with $0 < \delta_k(t) \le \delta_j^k$, $0 < r_k(t) \le r_j$ (resp. r_{i+1}^k for $j = i+1$) on $[2^{j-1}, 2^j]$ for $j = 2, \ldots, i+1$ resp. $[0, 2]$ for $j = 1$ and Ricci flows with surgery $\mathcal{M}_k$ (defined on the time interval $[0, T_k)$) that satisfy (i)-(iii) for $\delta_k(t)$ and $r_k(t)$. Moreover, we find points $(x_k, t_k) \in \mathcal{M}_k$ corresponding to the point (x, t_0) in the preceding paragraphs that do not satisfy $\mathrm{CNA}(2r_{i+1}^k, \frac{\varepsilon}{2}, \frac{E_0}{2})$. However, $\mathrm{CNA}(\frac{1}{2}r_{i+1}^k, 2\varepsilon, 2E_0)$ holds on $\mathcal{M}_k$ at times $[0, t_k]$. Observe that $t_k \le \min\{2^{i+1}, T_k\}$ and $t_k \ge 2^i$ for $i > 0$ resp. $t_k \ge \frac{1}{4}$ for large k and $i = 0$.

Let $Q_k := S(x_k, t_k) \ge \frac{1}{(2r_{i+1}^k)^2} \to \infty$, shift $\mathcal{M}_k$ in time by $-t_k$, parabolically rescale by $Q_k^{1/2}$ and restrict to nonpositive times to get $\mathcal{M}'_k$. Assume that $\mathcal{M}'_k$ is defined on $[-T'_k, 0]$. For the points $(x'_k, 0) \in \mathcal{M}'_k$ corresponding to the (x_k, t_k) we have $S(x'_k, 0) = 1$. Furthermore, $T'_k \to \infty$ and the curvature on $\mathcal{M}'_k$ is φ_k-positive with $\varphi_k \to 0$. Observe that since $Q_k \ge \frac{1}{(2r_{i+1}^k)^2}$ the $\mathcal{M}'_k$ satisfy the canonical neighborhood assumptions $\mathrm{CNA}(\frac{1}{4}, 2\varepsilon, 2E_0)$ at all non-presurgery points, but $(x'_k, 0)$ does not satisfy $\mathrm{CNA}(1, \frac{\varepsilon}{2}, \frac{E_0}{2})$. Moreover, we have κ_{i+1}-noncollapsedness everywhere.

By Lemma 7.4.3 (i) and Proposition 6.2.4 we can exclude surgery points on larger and larger balls $B_{d_k}(x'_k, 0)$ and even on times slightly before 0 (depending on the distance to x'_k). So by Lemma 6.2.2 and Shi's estimates we have smooth convergence

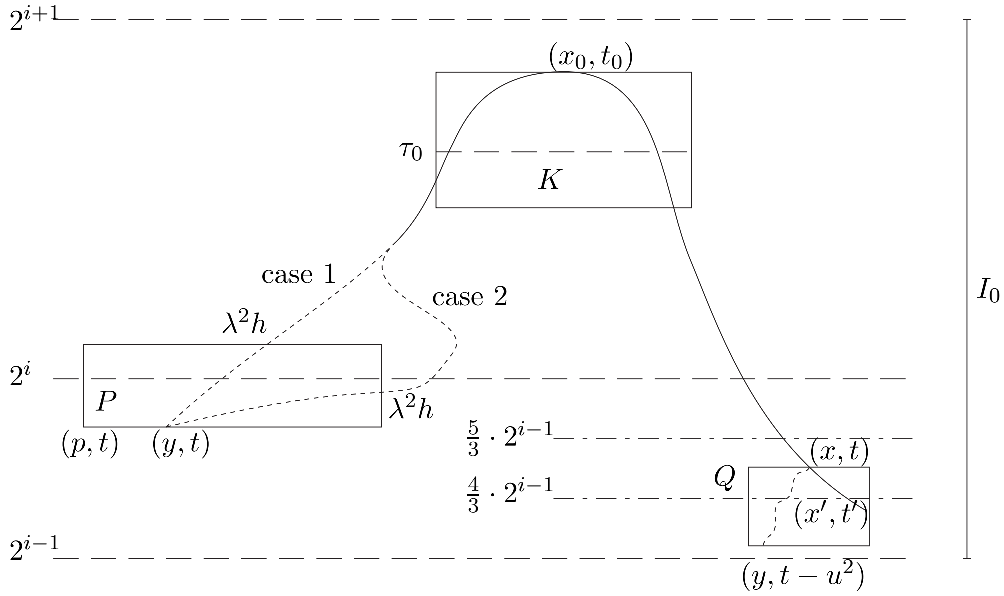
$$(B_{d_k}(x'_k, 0), (x'_k, 0)) \xrightarrow[k \to \infty]{} (M_\infty(0), (x_\infty, 0)). \tag{7.5}$$

Analogous to the proof of Proposition 6.3.1 we conclude that $M_\infty(0)$ has bounded, nonnegative sectional curvature. Let $T \ge 0$ be maximal with the property that there are sequences $d_k \to \infty$ and $\tau_k \to T$ such that the $P(x'_k, 0, d_k, -\tau_k)$ are non-singular and the Ricci flows on these parabolic neighborhoods converge in (7.5) to some Ricci flow $M_\infty \times (-T, 0]$. By Proposition 6.3.1, the scalar curvature on $M_\infty \times (-T, 0]$ can be bounded by some constant K. Assume that $T < \infty$. Then for $b := \frac{1}{4}\beta K^{-1}$ and any $a < \infty$ we can bound the scalar curvature on $P(x'_k, 0, a, -T + b)$ by $2K$ for large k and we may exclude surgery points there. Thus by Lemma 7.4.3 (ii) we can even exclude surgery points on the parabolic neighborhoods $P(x'_k, 0, a, -T - b)$ for large k (we first have to rescale back and work on $\mathcal{M}$) and by Proposition 6.3.1 we can extend the convergence (7.5) to a convergence of Ricci flows on the interval $(-T - b, 0]$ contradicting the maximality of T.

So there are sequences $d_k \to \infty$ and $\tau_k \to \infty$ such that

$$(P(x'_k, 0, d_k, 0), (x'_k, 0)) \xrightarrow[k \to \infty]{} (M_\infty \times (-\infty, 0], (x_\infty, 0)).$$

It is easy to see, that $M_\infty \times (-\infty, 0]$ is an orientable κ_{i+1}-solution that is not a higher spherical space form. Now use Theorems 5.4.13 and 5.4.11 to find that $(x_\infty, 0)$ is the center of a strong $\frac{\varepsilon}{4}$-neck or $(\frac{\varepsilon}{4}, \frac{E_0}{4})$-cap. Hence $(x'_k, 0)$ and then also (x_k, t_k) must have been the center of a strong $\frac{\varepsilon}{2}$-neck or an $(\frac{\varepsilon}{2}, \frac{E_0}{2})$-cap for large k (recall the definition of E_0). Moreover we conclude that property (A) by Corollary 5.2.9 and property (B) in the definition of the canonical neighborhood assumptions hold at $(x_k', 0)$ and thus also at (x_k, t_k) for large k. This contradicts the choice of (x_k, t_k). $\square$



Proof of Lemma 7.5.2. We want to show noncollapsedness at some point $(x_0, t_0) \in \mathcal{M}$ with $2^i \leq t_0 \leq 2^{i+1}$ resp. $0 \leq t_0 \leq 2^{i+1}$ for $i = 0$. Let $0 < r \leq \sqrt{t_0}$ and assume that $P(x_0, t_0, r, -r^2)$ is non-singular and we have $\|R\| \leq \frac{1}{r^2}$ there. We will estimate the volume of $B(x_0, t_0, r)$ from below.

In order to carry out the argument, we will use the tools developed in chapter 4 and the proof is similar to the proof of the No Local Collapsing Theorem 4.2.4. The difficulty here is to take care that points in space-time may only be joined by broken $\mathcal{L}$-geodesics that hit surgery points. We have to analyze these geodesics and conclude that their $\mathcal{L}$-length can be assumed to be arbitrarily large if δ is chosen sufficiently small. On the other hand, we are just able to control the δ-parameter on $[2^{i-1}, 2^{i+1}]$ resp. $[0,2]$ for $i = 0$ and $[0,4]$ for $i = 1$. So for $i > 1$ we are not able to use the noncollapsedness of the initial manifold.

At first, we consider the case in which $S(x_0, t_0)$ is large.

Noncollapsedness for $S(x_0, t_0) > \frac{1}{r_{i+1}^2}$. In this case the canonical neighborhood assumptions $\mathrm{CNA}(r_{i+1}, \varepsilon, E_0)$ apply at (x_0, t_0). So $\mathcal{M}$ is $\frac{1}{2}\kappa_0$-noncollapsed in (x_0, t_0).

For the rest of this proof we will assume that

$$S(x_0, t_0) \leq \frac{1}{r_{i+1}^2}.$$

Existence of $\mathcal{L}$-geodesics. Consider (x_0, t_0) as a basepoint and introduce the time parameter $\tau := t_0 - t$ as in chapter 4. Let $(y, t) \in \mathcal{M}$ be some point with $t < t_0$ that is not a presurgery point. Denote the surgery times of $\mathcal{M}$ between t and t_0 by $t_1', \ldots, t_l'$. Consider piecewise smooth curves $\gamma : [0, t_0 - t] \to \mathcal{M}$ in space-time (i.e. $\gamma(\tilde{\tau}) \in \mathcal{M}(t_0 - \tilde{\tau})$) joining (x_0, t_0) with (y, t) and let $\mathcal{L}(\gamma)$ be their $\mathcal{L}$-length. Let $L(y, t)$ be the infimum over all those $\mathcal{L}$-lengths.

If $(t_1, t_2) \subset [0, t_0]$ is a time interval that does not contain surgery times, we know that for any two points $(y_a, t_a), (y_b, t_b) \in \mathcal{M}$ with $t_1 < t_a < t_b < t_2$ there is a minimizing $\mathcal{L}$-geodesic joining these points. So by a limiting argument we conclude that this is still true if we require that $t_1 \leq t_a < t_b \leq t_2$ and that (y_a, t_a) is a non-presurgery and (y_b, t_b) a non-postsurgery point. Moreover, the $\mathcal{L}$-length of this minimizing $\mathcal{L}$-geodesic varies continuously with its endpoints. Obviously, if we replace an arbitrary piecewise smooth curve $\gamma : [0, t_0 - t] \to \mathcal{M}$ in space-time that joins (x_0, t_0) with (y, t) by a broken $\mathcal{L}$-geodesic $\gamma' : [0, t_0 - t] \to \mathcal{M}$ in space-time with breaking times $t'_1, \ldots, t'_l$ and $\gamma'(t'_i) = \gamma(t'_i)$, we have $\mathcal{L}(\gamma') \leq \mathcal{L}(\gamma)$. By varying the breaking points we see that there is an $\mathcal{L}$-minimizing piecewise smooth curve $\gamma : [0, t_0 - t] \to \mathcal{M}$ between (x_0, t_0) and (y, t). Moreover, γ is a broken $\mathcal{L}$-geodesic and all breaking points (if there are any) lie on surgery points. If γ does not hit any surgery points, we call γ *admissible*. Observe that any admissible broken $\mathcal{L}$-geodesic is smooth.

Estimates of $\mathcal{L}$- and $\mathcal{L}^+$-length. For a broken curve $\gamma : [0, \tau] \to \mathcal{M}$ in space time we define

$$\mathcal{L}^+(\gamma) = \int_0^\tau \sqrt{\tilde{\tau}}(\|\dot{\gamma}\|^2 + S^+(\gamma))\mathrm{d}\tilde{\tau},$$

where $S^+ := \max\{0, S\}$. By the 1-positive curvature assumption $S \geq -3$, so we get

$$\mathcal{L}^+ - 2\tau^{3/2} = \mathcal{L}^+ - 3\int_0^\tau \sqrt{\tilde{\tau}}\mathrm{d}\tilde{\tau} \leq \mathcal{L} \leq \mathcal{L}^+. \tag{7.6}$$

Estimate of the $\mathcal{L}$-lengths of non-admissible curves. From now on consider $\mathcal{L}$-geodesics with basepoint (x_0, t_0) that do not exist outside the time interval $I_0 := [2^{i-1}, 2^{i+1}]$ for $i > 1$, $I_0 := [0, 2]$ for $i = 0$ or $I_0 := [0, 4]$ for $i = 1$. Let $\mathcal{L}_0 < \infty$ be a given constant. We want to show that for δ sufficiently small (depending only on $\mathcal{L}_0$ and the parameters r_i) every such non-admissible $\mathcal{L}$-geodesic γ has $\mathcal{L}(\gamma) \geq \mathcal{L}_0$.

Let ΔT be the length of I_0. We have $\Delta T = 2^{i+1} - 2^{i-1} = 3 \cdot 2^{i-1}$ for $i > 1$, $\Delta T = 2$ for $i = 0$ and $\Delta T = 4$ for $i = 1$. Set $\mathcal{L}_0^+ := \mathcal{L}_0 + 2\Delta T^{3/2}$. By (7.6) it suffices to prove that $\mathcal{L}^+(\gamma) \geq \mathcal{L}_0^+$ for sufficiently small δ.

Let $\gamma : [0, \tau] \to \mathcal{M}$ in space-time be a non-admissible minimal broken $\mathcal{L}$-geodesic joining (x_0, t_0) with (y, t), $t \in I_0$. Without loss of generality we may assume that $(y, t) = \gamma(\tau)$ is the first surgery point on γ. Let (p, t) be the tip of the corresponding surgery.

Since $S(x_0, t_0) \leq \frac{1}{r_{i+1}^2}$, we get by Lemmas 6.2.1 and 6.2.2 that there is a universal constant $v > 0$ such that $S < \frac{16}{r_{i+1}^2}$ on $K := P(x_0, t_0, v r_{i+1}, -v^2 r_{i+1}^2)$. So for sufficiently small δ the parabolic neighborhood K is non-singular because surgery points at times I_0 can be assumed to have scalar curvature greater than $\frac{1}{2\delta^2 r_i^2}$ or $\frac{1}{2\delta^2 r_i^2}$. The 1-positive curvature condition gives us a bound for $\|R\|$ on K. So there is a universal bilipschitz bound C for the distortion of the metric on K.

Assume that γ leaves $B(x_0, t_0, v r_{i+1})$ before time $\tau_0 := \min\left\{\frac{v^4 r_{i+1}^4}{4C^2(\mathcal{L}_0^+)^2}, v^2 r_{i+1}^2\right\}$. Then we have

$$\mathcal{L}^+(\gamma) \geq \int_0^{\tau_0} \sqrt{\tilde{\tau}}\|\dot{\gamma}\|^2\mathrm{d}\tilde{\tau} \geq \int_0^{\tau_0} \sqrt{\tilde{\tau}}C^{-1}\|\dot{\gamma}\|_0^2\mathrm{d}\tilde{\tau}$$

Reparameterizing via $\tilde{\tau} = s^2$ gives (see (4.3))

$$\int_0^{\tau_0} \sqrt{\tilde{\tau}}C^{-1}\|\dot{\gamma}\|_0^2\mathrm{d}\tilde{\tau} = \frac{1}{2}C^{-1}\int_0^{\sqrt{\tau_0}} \left\|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\right\|_0^2 \mathrm{d}s \geq \frac{1}{2}C^{-1}\frac{v^2 r_{i+1}^2}{\sqrt{\tau_0}} \geq \mathcal{L}_0^+.$$

So assume now that $\gamma|_{[0,\tau_0)} \subset K$. Since there are no surgery points in K, this implies $\tau \geq \tau_0$.

Let $A < \infty$, $\theta, \varphi > 0$ be constants that will be specified later. By Lemma 7.4.1 we find that for δ sufficiently small there is some $0 \leq \sigma \leq 1 - \theta$ such that there is a standard solution $M_{\mathrm{stan}} \times [0, 1)$ with tip p_{stan} and a φ-homothety with some scaling factor λ

$$M_{\mathrm{stan}} \times [0, 1) \supset B_A(p_{\mathrm{stan}}, 0) \times [0, \sigma) \xrightarrow{\Phi} \mathcal{M} \qquad \text{with} \qquad \Phi(p_{\mathrm{stan}}, 0) = (p, t).$$

Denote $P := \overline{\operatorname{Im}\Phi}$. Furthermore, if $\sigma < 1-\theta$ no point of P survives past time $t+\lambda^2\sigma$.

For φ sufficiently small we have by Theorem 7.3.2 (c)

$$S > \frac{1}{2\delta^2 r_{i+1}^2} \quad \text{or} \quad S > \frac{1}{2\delta^2 r_i^2} \qquad \text{on} \qquad P.$$

So choosing δ sufficiently small, we may assume that $P \cap K = \emptyset$.

Observe that this implies that γ has to enter P after time τ_0. Let $t+\lambda^2 h$ be the smallest time in which γ enters P. Consider now γ to be parameterized by time t (rather than backward time τ) and denote by $\gamma' : [0,h) \to M_{\text{stan}} \times [0,1)$ the pullback of $\gamma|_{[t,t+\lambda^2 h)}$ under Φ. We have for φ sufficiently small

$$\begin{aligned}\mathcal{L}^+(\gamma) &\geq \int_t^{t+\lambda^2 h^2} \sqrt{t_0-\tilde{t}}\left(\|\dot\gamma\|^2 + S^+(\gamma(\tilde t),\tilde t)\right)\mathrm{d}\tilde t \\ &\geq \sqrt{\tau_0}\int_t^{t+\lambda^2h^2}\left(\|\dot\gamma\|^2 + S^+(\gamma(\tilde t),\tilde t)\right)\mathrm{d}\tilde t \geq \frac{\sqrt{\tau_0}}{2}\int_0^h\left(\|\dot\gamma'\|^2+S^+(\gamma'(\tilde t),\tilde t)\right)\mathrm{d}\tilde t\end{aligned}$$

where the quantities under the last integral sign are taken on $M_{\text{stan}}\times[0,1)$. Set $\theta := \exp(-\frac{2}{c\sqrt{\tau_0}}\mathcal{L}_0^+)$ where c is the constant from Theorem 7.3.2 (b). There are two cases

case 1: γ enters P at its final time slice. Then we have $h=\sigma=1-\theta$. Furthermore by Theorem 7.3.2 (b)

$$\int_0^{1-\theta}\left(\|\dot\gamma'\|^2+S^+(\gamma'(\tilde t),\tilde t)\right)\mathrm{d}\tilde t \geq \int_0^{1-\theta}\frac{c}{1-\tilde t}\mathrm{d}\tilde t = -c\log\theta = \frac{2}{\sqrt{\tau_0}}\mathcal{L}_0^+.$$

case 2: γ enters P "before" its final time slice. Recall the surgery process. We find that there is a universal $d_0<\infty$ such that $\operatorname{dist}_0(p_{\text{stan}},\gamma'(0)) < d_0$. By Theorem 7.3.2 (d) there is a universal bound for the curvature on $M_{\text{stan}}\times[0,1-\theta)$ and thus we find a bilipschitz bound C for the distortion of the metric (observe that C is independent of the standard solution $M_{\text{stan}}\times[0,1)$). Let $A := \left(\frac{2hC^2\mathcal{L}_0^+}{\sqrt{\tau_0}}\right)^{1/2} + d_0$. Then

$$\begin{aligned}\int_0^h(\|\dot\gamma'\|_{\tilde t}^2+S^+)\mathrm{d}\tilde t &\geq \int_0^h\|\dot\gamma'\|_{\tilde t}^2\mathrm{d}\tilde t \geq \frac{1}{h}\left(\int_0^h\|\dot\gamma'\|_{\tilde t}\mathrm{d}\tilde t\right)^2 \\ &\geq \frac1h\left(\int_0^h C^{-1}\|\dot\gamma'\|_0\mathrm{d}\tilde t\right)^2 \geq \frac{(A-d_0)^2}{hC^2} \geq \frac{2}{\sqrt{\tau_0}}\mathcal{L}_0^+.\end{aligned}$$

So in both cases $\mathcal{L}^+(\gamma)\geq\mathcal{L}_0^+$.

Finding a point of low $\mathcal{L}$-distance and low curvature Let $(y,t)\in\mathcal{M}$ with $t<t_0$. We say that (y,t) is admissible if there is a $\theta>0$ and an admissible minimizing $\mathcal{L}$-geodesic $\gamma:[0,t_0-t+\theta]\to\mathcal{M}$ in space-time (parameterized by backwards time) such that $\gamma(0)=(x_0,t_0)$ and $\gamma(t_0-t)=(y,t)$. Let $\mathcal{M}_{\text{adm}}\subset\mathcal{M}$ be the (open) set of admissible points.

Set $\bar L(y,t) := 2\sqrt{t_0-t}L(y,t)$ and $l(y,t) := \frac{1}{2\sqrt{t_0-t}}L(y,t)$. Recall from (4.11) that we have in the barrier sense

$$-\frac{\partial}{\partial t}\bar L + \triangle\bar L \leq 6 \tag{7.7}$$

at all admissible points.

Set $t' := 2^{i-1}+\frac13 2^{i-1} = \frac43 2^{i-1}$ for $i>1$ or $t' := \frac14$ for $i=0,1$. We will first show that there is a point $x'\in\mathcal{M}_{\text{adm}}(t')$ such that $l(x',t')\leq\frac32$. Choose δ so small that any non-admissible $\mathcal{L}$-geodesic that does not exist outside the time interval I_0 has $\mathcal{L}$-length

$> 5\sqrt{\Delta T}$. We conclude that for any point $(y,t) \in \mathcal{M} \setminus \mathcal{M}_{\mathrm{adm}}$ with $t \in I_0$ we have $\bar{L}(y,t) > 8(t_0 - t) + 2\sqrt{\Delta T}\sqrt{t_0 - t}$. Assume that $\bar{L} > (6+\varphi)(t_0 - t') + \varphi\sqrt{t_0 - t'}$ on $\mathcal{M}_{\mathrm{adm}}(t')$ for some $\varphi > 0$. Let $\tilde{t}$ be the supremum over all times $\tilde{t} \in [t', t_0]$ such that $\bar{L} > (6+\varphi)(t_0 - \tilde{t}) + \varphi\sqrt{t_0 - \tilde{t}}$ on $\mathcal{M}_{\mathrm{adm}}(\tilde{t})$. Then for $\varphi < 2\sqrt{\Delta T}$

$$\bar{L} \geq (6+\varphi)(t_0 - \tilde{t}) + \varphi\sqrt{t_0 - \tilde{t}}$$

on $\mathcal{M}(\tilde{t})$. If $\tilde{t} < t_0$, there is a point $\tilde{x} \in \mathcal{M}(\tilde{t})$ where equality holds. So $(\tilde{x}, \tilde{t})$ must be admissible and $\triangle \bar{L}(\tilde{x}, \tilde{t}) \geq 0$. From (7.7) we find $\frac{\partial}{\partial t}|_{t=\tilde{t}}\bar{L}(\tilde{x}, t) \geq -6$ contradicting the choice of $\tilde{t}$. On the other hand $\tilde{t} = t_0$ implies $L(x_0, t'') > \frac{\varphi}{2}$ for all $t'' < t_0$, a contradiction. Thus, we conclude that for any $\varphi > 0$ there is a point $x'' \in \mathcal{M}(t')$ with $\bar{L}(x'', t') \leq (6+\varphi)(t_0 - t') + \varphi\sqrt{t_0 - t'}$. So there is also a point $x' \in \mathcal{M}(t')$ with $\bar{L}(x', t') \leq 6(t_0 - t')$ which has to be admissible then. The last inequality implies

$$l(x', t') \leq \frac{3}{2}.$$

Let $\gamma : [0, t_0 - t' + \theta) \to \mathcal{M}$ in spacetime be an admissible minimizing $\mathcal{L}$-geodesic with $\gamma(0) = (x_0, t_0)$ and $\gamma(t_0 - t') = (x, t')$ for some $\theta > 0$. Note that all points on γ are admissible points.

For $i = 0, 1$ we have $S(x', t') < 2$ by the normalized initial conditions and Corollary 2.5.5. In this case set $(x, t) := (x', t')$. For the rest of this step assume $i > 1$.

Assume that at every point on $\gamma|_{[t_0 - \frac{5}{3}2^{i-1}, t_0 - \frac{4}{3}2^{i-1}]}$ we have $S > 256$. The 1-positive curvature condition implies $S > -3$ everywhere. So we get

$$\begin{aligned} L(x', t') &= \int_0^{t_0 - \frac{4}{3}\cdot 2^{i-1}} \sqrt{\tau}\left(\|\dot{\gamma}\|^2 + S\right) \mathrm{d}\tau > -3\int_0^{t_0 - \frac{5}{3}2^{i-1}} \sqrt{\tau}\mathrm{d}\tau + 256\int_{t_0 - \frac{5}{3}2^{i-1}}^{t_0 - \frac{4}{3}2^{i-1}} \sqrt{\tau}\mathrm{d}\tau \\ &\geq -3\int_0^{2^{i+1}} \sqrt{\tau}\mathrm{d}\tau + 2^8\int_{\frac{1}{3}2^{i-1}}^{\frac{2}{3}2^{i-1}} \sqrt{\tau}\mathrm{d}\tau > -2\cdot 2^{\frac{3}{2}(i+1)} + \frac{2^8}{3^{\frac{3}{2}}}2^{\frac{3}{2}i - \frac{3}{2}} \geq -2\cdot 2^{\frac{3}{2}(i+1)} + 2^{8 + \frac{3}{2}i - 4}. \end{aligned}$$

Combining this with the upper bound on $L(x', t')$ we get

$$6 \cdot 2^{\frac{1}{2}i} > 3\sqrt{2^{i+1}} \geq 3\sqrt{t_0 - t'} \geq L(x', t') > -2^{\frac{3}{2}i + \frac{5}{2}} + 2^{\frac{3}{2}i + 4}.$$

So

$$6 > -2^{i + \frac{5}{2}} + 2^{i+5} > -32 + 64,$$

a contradiction.

Thus there is an admissible point $(x, t) \in \mathcal{M}$ with $t \in [\frac{1}{3}2^{i-1} + 2^{i-1}, \frac{2}{3}2^{i-1} + 2^{i-1}]$ that has $S(x, t) \leq 256$ and

$$\begin{aligned} L(x, t) &= \int_0^{t_0 - t} \sqrt{\tau}\left(\|\dot{\gamma}\|^2 + S\right) \mathrm{d}\tau = \mathcal{L}(\gamma|_{[0, t_0 - t']}) - \int_{t_0 - t}^{t_0 - t'} \sqrt{\tau}\left(\|\dot{\gamma}\|^2 + S\right) \mathrm{d}\tau \\ &\leq 3\sqrt{t_0 - t'} - \int_{t_0 - t}^{t_0 - t'} \sqrt{\tau} S \mathrm{d}\tau \leq (3 + 3\cdot 2^{i-2})\sqrt{t_0 - t'} \leq 2^{\frac{3}{2}i + 2}. \end{aligned}$$

Estimating the reduced volume of the admissible points from below Analogous to (4.14) we define

$$\tilde{V}_{\mathrm{adm}}(\tau) := \int_{\mathcal{M}_{\mathrm{adm}}(t_0 - \tau)} \tau^{-n/2} e^{-l(\cdot, \tau)} \mathrm{d}\mu_\tau.$$

As in chapter 4 we can show that $\tilde{V}_{\mathrm{adm}}(\tau)$ is nonincreasing in τ. We want to estimate $\tilde{V}_{\mathrm{adm}}(r^2)$ from below.

Consider first the case $i > 1$. Replacing r by $\frac{r}{\sqrt{12}}$ we may assume that $r^2 \leq \frac{1}{3}2^{i-1}$ (observe that $r^2 \leq 2^{i+1}$). Since $S(x,t) \leq 256$, we conclude by Lemmas 6.2.1, 6.2.2 and the 1-positive curvature condition that

$$S < \frac{1}{u^2} \qquad \text{on} \qquad Q := P(x,t,u,-u^2)$$

for some universal $0 < u < 1$. So if we assume δ to be sufficiently small, there are no surgery points on Q. Thus by the κ_i-noncollapsedness we have $\mathrm{vol}_t\, B_u(x,t) \geq \kappa_i u^3$. Let C be a bilipschitz bound for the distortion of the Riemannian metric on P. We find $\mathrm{vol}_{t-u^2}\, B_u(x,t) \geq C^{-3/2}\kappa_i u^3$. Now we want to bound l from above on $B_u(x,t) \times \{t-u^2\}$. Observe that $L(x,t) \leq 2^{\frac{3}{2}i+2}$. For $y \in B_u(x,t)$ choose a minimizing time t geodesic $\gamma' : [t_0-t, t_0-t+u^2] \to B_u(x,t)$ between x and y. We have

$$\begin{aligned} L(y, t_0-t+u^2) < L(x, t_0-t) + \mathcal{L}(\gamma') \leq 2^{\frac{3}{2}i+2} + \int_{t_0-t}^{t_0-t+u^2} \sqrt{\tau}(\|\dot\gamma'\|^2 + S)\mathrm{d}\tau \\ < 2^{\frac{3}{2}i+2} + u^2\sqrt{2^{i+1}}(Cu^{-2} + u^{-2}). \end{aligned}$$

So for sufficiently small δ the points on $B_u(x,t) \times \{t-u^2\}$ are admissible. Furthermore, since $t_0 - t + u^2 \geq \frac{1}{3}2^{i-1}$, we find

$$l(y, t_0-t+u^2) < \frac{1}{2\sqrt{2^{i-1}/3}}\left(2^{\frac{3}{2}i+2} + \sqrt{2^{i+1}}(C+1)\right) =: E.$$

and thus

$$\tilde V_{\mathrm{adm}}(r^2) \geq \tilde V_{\mathrm{adm}}(t_0-t+u^2) \geq (2^{i+1})^{-3/2} e^{-E} \cdot C^{-3/2}\kappa_i u^3.$$

In the cases $i = 0, 1$ the estimation of $\tilde V_{\mathrm{adm}}(r^2)$ is even easier. Observe that we have control over the scalar curvature at times $[0,t] = [0, \frac{1}{4}]$ and that there are no surgery times before time $\frac{1}{2}$. Moreover, $\mathcal{M}(0)$ is 1-noncollapsed on scales < 1. The reasoning is then almost the same as in the proof of the No Local Collapsing Theorem 4.2.4.

Noncollapsedness at (x_0, t_0) Analogous to (4.4) let

$$D^\tau_{\mathrm{adm}} := \left\{ v \in T_{x_0}M \;:\; \begin{array}{l} \gamma_v : [0, \tau+\delta] \to M \text{ is } \mathcal{L}\text{-minimizing} \\ \text{and admissible for some } \delta > 0 \end{array} \right\}$$

be the *admissible domain.* It is easy to see that

$$\tilde V_{\mathrm{adm}}(\tau) = \int_{D^\tau_{\mathrm{adm}}} \tau^{-n/2} e^{-l_\tau(v)} J_\tau(v)\mathrm{d}v.$$

Now we can go through the steps of the proof of Lemma 4.2.3 to find that the lower bound on $\tilde V_{\mathrm{adm}}(r^2)$ implies that there is a constant $\kappa_{i+1} > 0$ such that $\mathrm{vol}_{t_0}\, B_r(x_0,t_0) \geq \kappa_{i+1} r^3$ (we just have to consider the set of admissible points instead of the manifold resp. the admissible domain instead of the domain). $\square$

Chapter 8

Outlook

So far we have proven that a Ricci flow with surgery $\mathcal{M}$ can always be constructed in the way described in section 7.2 if we start with a manifold $M = \mathcal{M}(0)$ having normalized geometry such that surgery times do not accumulate. So in a finite time interval there are only finitely many surgery times. It is now important to analyze the long time behaviour of $\mathcal{M}$.

If the manifold $\mathcal{M}(t)$ is empty for some large t, we say that the Ricci flow goes *extinct* after finite time. In this case it is easy to see that M is a connected sum of space forms and a certain number of copies of $S^2 \times S^1$. Using Corollary 2.5.5, we can show that extinction in finite time is always guarateed if M has positive scalar curvature. Note hereby that after a surgery the minimum of the scalar curvature does not decrease.

So we have proved

Theorem 8.0.1. *Every compact orientable Riemannian* 3*-manifold of positive scalar curvature is a connected sum of space forms and finitely many copies of* $S^2 \times S^1$.

We will now describe how the Poincaré Conjecture can be shown using Ricci flow with surgery. Assume therefore M to be simply connected. Then M is already homotopy equivalent to S^3 (see [Hat, Prop. 3.7]). By Theorem 1.5 in [Hat] M admits a unique decomposition $M = P_1 \# \ldots \# P_m$ as a connected sum of closed 3-manifolds $P_1, \ldots, P_m$ that are *irreducible* (i.e. every embedded 2-sphere bounds an embedded ball) such that none of these manifolds is diffeomorphic to S^3 if $m > 1$. From van Kampen's Theorem we conclude that all the manifolds $P_1, \ldots, P_m$ are simply connected. If these manifolds are 3-spheres, M is a 3-sphere as well. This shows that in order to prove the Poincaré Conjecture, we may assume M to be irreducible. Note that under this assumption the surgeries in $\mathcal{M}$ do not change the diffeomorphism type of the manifold. Looking closer at the surgery process we find that at any surgery time for any $\xi > 0$ the metric on the postsurgery time slice is at most $1 + \xi$ times the metric on the presurgery time slice for an appropriate identification of the pre- and postsurgery time slice.

If we could prove that the Ricci flow $\mathcal{M}$ goes extinct after finite time, we would get that M can only be a 3-sphere and this would prove the Poincaré conjecture. There are two arguments for this extinction result known to the author:

(A) The first argument is due to Perelman. Let ΛM be the space of all loops in M. Since M is a homotopy sphere, we have $\pi_3 M \cong \mathbb{Z}$. Observe now that $\pi_3 M \cong \pi(\Lambda M, M)$ by the following reason: It is easy to see that $S^3 \approx S^1 \times \mathbb{D}^2 / \sim$ where $(x, y) \sim (x', y)$ if $y \in \partial \mathbb{D}^2$. So any map $\beta : S^3 \to M$ can be seen as a map $(\mathbb{D}^2, \partial \mathbb{D}^2) \to (\Lambda M, M)$ and vice versa. Fix a nontrivial class $\alpha \in \pi(\Lambda M, M)$. Now for any continuously differentiable $c \in \Lambda M$ define $A_t(c)$ to be the infimum over the areas of all discs in $\mathcal{M}(t)$ whose boundary is c. Furthermore, for any continuously differentiable map

$\beta : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (\Lambda M, M)$ set

$$A_t(\beta) := \sup_{x \in \mathbb{D}^2} A_t(\beta(x)).$$

Finally, define

$$A_t(\alpha) := \inf_{\beta \in \alpha} A_t(\gamma')$$

where the infimum is only taken over continuously differentiable β. In [Per3] it is shown[1] that

$$\dot{A}_t(\alpha) \leq -2\pi - \frac{1}{2} \min_{\mathcal{M}(t)} S A_t(\alpha). \tag{8.1}$$

Moreover, if $t > 0$ is a surgery time, the fact that the metric shrinks after a surgery implies $A_{t^+}(\alpha) \leq A_{t^-}(\alpha)$. Now observe that since $S \geq -3$ on $\mathcal{M}(0)$ and since the minimum of the scalar curvature in nondecreasing under a surgery, Corollary 2.5.5 gives $S \geq -\frac{3}{1+2t}$ on $\mathcal{M}(t)$. Combining this with (8.1), we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{A_t(\alpha)}{(\frac{1}{2} + t)^{3/4}} \right) \leq -\frac{2\pi}{(\frac{1}{2} + t)^{3/4}}.$$

Since the right hand side is non-integrable at infinity, this implies that if $\mathcal{M}$ did not go extinct after finite time, we must have $A_t(\alpha) < 0$ for large t. A contradiction.

Note that the proof of (8.1) requires some non-basic analytical tools such as the existence of minimal area disc with prescribed boundary, an analysis of their boundary behaviour and the theory of the curve shortening flow.

(B) The second argument was developed by Colding and Minicozzi. Fix a nontrivial class $\alpha \in \pi_3 M \cong \pi_1(\mathcal{C}(S^2, M), M)$. We understand this identification in the following sense: any continuous map $\beta : S^3 \to M$ can be seen as a map $\beta : ([0,1], \partial[0,1]) \to (\mathcal{C}(S^2, M), M)$ that is a map $\beta : [0,1] \times S^2 \to M$ with the property that $\beta(0, \cdot)$ and $\beta(1, \cdot)$ are constant. Define now

$$W_t(\alpha) := \inf_{\beta \in \alpha} \sup_{s \in [0,1]} E_t(\beta(s, \cdot))$$

where E_t denotes the energy and the infimum is taken over all β such that $\beta(s, \cdot)$ is continuous and L_1^2 for all $s \in [0,1]$. It can now be shown that

$$\dot{W}_t(\alpha) \leq -4\pi - \frac{1}{2} \min_{\mathcal{M}(t)} S W_t(\alpha). \tag{8.2}$$

Moreover it is easy to see that if t is a surgery time, we have $W_{t^+}(\alpha) \leq W_{t^-}(\alpha)$. Analogous to (A) we can conclude extinction in finite time.

A proof of (8.2) can be found in [CM1]. However, the proof assumes a profound knowledge about results in the theory of minimal surfaces and the construction of so-called min-max surfaces. Recently, the authors of [CM1] have published an update [CM2] which promises to explain these prerequisites.

In order to prove the Geometrization Conjecture, we have to analyze the behaviour of $\mathcal{M}$ for $t \to \infty$. Observe that if M is an arbitrary oriented closed 3-manifold, extinction may not take place. In [Per2, Sec 6,7] and [KL, Sec 80ff] it is shown that for large t the time slice $\mathcal{M}(t)$ is the union of two sets called the thick part $\mathcal{M}_{\text{thick}}(t)$ and the thin part $\mathcal{M}_{\text{thin}}(t)$. The thick part gets closer and closer to a hyperbolic metric and points x

[1] Observe that in [Per3] Perelman proves the *Elliptization Conjecture* which is more general than the Poincaré Conjecture.

in the thin part have the following property: there is a radius $\rho < \operatorname{diam} \mathcal{M}(t)$ such that $K \geq -\rho^{-2}$ on $B_\rho(x,t)$ and $\frac{1}{\rho^3} \operatorname{vol}_t B_\rho(x,t)$ is smaller than some constant that goes to 0 as $t \to \infty$. So $\mathcal{M}_{\text{thick}}(t)$ is a hyperbolic manifold for large t. Furthermore, it is possible to show that its the cuspidal tori are incompressible in $\mathcal{M}(t)$.

Now it remains to prove that $\mathcal{M}_{\text{thin}}(t)$ is essentially a certain type of manifold which admits a decomposition as required in the Geometrization Conjecture. Unfortunately, there is no complete proof of this fact in the literature by now.

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